A NOTE ON THE BEHAVIOR OF CERTAIN AUTOMORPHIC FUNCTIONS AND FORMS NEAR THE REAL AXIS

By David Rosen

Let $\Gamma(\lambda)$ denote the group of linear fractional transformations of the complex plane onto itself:

$$V(z) = (az + b)/(cz + d), \qquad ad - bc = 1, \tag{1}$$

where the coefficients a, b, c, d, are real numbers. $V(z)$ is generated by

$$T(z) = -1/z, \qquad S(z) = z + \lambda, \tag{2}$$

where $\lambda = 2 \cos \pi/q$, $q =$ integer ≥ 3, when $\lambda < 2$, and for all real values of λ, when $\lambda > 2$. For these values of λ, Hecke [3] has shown that $\Gamma(\lambda)$ is Fuchsian, that is $\Gamma(\lambda)$ is a properly discontinuous group for each λ. Accordingly we shall take as fundamental domain, the region in the upper-half of the complex plane defined by $| R(z) | \leq \lambda/2$, $| z | \geq 1$.

The purpose of this note is to describe the behavior of an automorphic function and form associated with $\Gamma(\lambda)$ near the real axis. The method is an extension of that employed by Epstein and Lehner [5] in their unpublished investigations for the case $\lambda = 1$, and by Cohn [1]. The modular case and the case $\lambda = 2$ are omitted from the discussion.

The limit points of $\Gamma(\lambda)$ on the principal circle have been characterized arithmetically [6] by means of certain continued fractions, called λ-fractions, which have the form

$$(r_0\lambda\ ,\ \epsilon_1/r_1\lambda\ ,\ \cdots\ ,\ \epsilon_n/r_n\lambda\ ,\ \cdots), \tag{3}$$

where the $r_i (i \geq 1)$ are positive integers, λ is defined as above, and $\epsilon_i = \pm 1$. It is shown in [6] that a limit point is a parabolic or cusp point if and only if its λ-fraction is finite, while a non-parabolic limit point has an infinite λ-fraction representation. Moreover, the λ-fraction is unique provided that Definition 1 in [6] is satisfied, and it is then called a reduced λ-fraction. In this paper all λ-fractions are reduced.

The main results depend on the following:

Lemma 1. *Let $z_n = x_n + iy_n$, $y_n > 0$ be a point in the upper half plane, and let $z'_n = (a_n z_n + b_n)/(c_n z_n + d_n) = x'_n + iy'_n$ be a substitution belonging to $\Gamma(\lambda)$. For every $c > 0$ and every real number α which has an infinite λ-fraction representation, there exist sequences of points $z_n = z_n(c)$ such that* (1) *$z_n(c) \to \alpha$ in a Stolz angle, and* (2) *$y'_n = c$ for all n.*

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