CONDITIONS ON THE REALIZATION OF PREDICTION BY MEASURES

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1. **Introduction.** Let x_t $(-\infty < t < \infty)$ be a second-order-stationary complex-valued stochastic process. For simplicity we shall suppose that the expectations $Ex_t \equiv 0$. The covariance function $\rho(t) = Ex_s\bar{x}_{s+t}$ has the spectral representation

$$\rho(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, dF(\lambda), \tag{1}$$

where F is a distribution function. We shall make the familiar regularity assumption

Condition A.

$$\int_{-\infty}^{\infty} \frac{|\ln S(\lambda)|}{1+\lambda^2}\, d\lambda < \infty,$$

where S indicates the derivative of F.

Given $\tau > 0$, it is known that there exists a random variable y_τ which minimizes the "prediction error"

$$\sqrt{E(x_\tau - y_\tau)^2} \tag{2}$$

over the class of all limits in the mean (— square) of linear combinations of x_t with $t \leq 0$.

In discussing the prediction problem in [4], N. Wiener also posed the following question: When can y_τ be realized as

$$y_\tau = \int_{-\infty}^{0} x_s\, dm^\tau(s), \tag{3}$$

where m_τ is a bounded, complex (Radon) measure on the Borel sets of $-\infty < t \leq 0$? Wiener demonstrated this to be the case in several examples where $S(\lambda)$ is of the form $S(\lambda) = N(\lambda^2)/D(\lambda^2)$, where N and D are polynomials, D being of degree one greater than N; and he pointed out the impossibility of such a realization under certain other conditions. It is to be expected, of course, that any condition on S implying differentiability of the sample functions (in some sense) would contra-indicate the possibility that (3) give a minimum prediction error.

The general problem of such "realizability", however, was left open; especially so for the cases where $S(\lambda)$ is not a ratio of polynomials.

More recently, A. Beurling [1] has formulated a necessary condition for such

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