THE BESSEL POLYNOMIALS

By W. A. Al-Salam

1. **Introduction.** The Bessel polynomials were introduced by Krall and Frink [14] in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solution of the differential equation

$$x^2y''(x) + (ax + b)y'(x) = n(n + a - 1)y(x) \tag{1.1}$$

where n is a positive integer and a and b are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function

$$\rho(x, \alpha) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha + n - 1)} \left(-\frac{2}{x}\right)^n. \tag{1.2}$$

Several other authors including Agarwal [1], Brafman [2], Burcnall [7], Carlitz [9], Dickinson [11], Grosswald [12], Rainville [15], and Toscano [16] have contributed to the study of the Bessel polynomials.

In this paper we propose to study further these polynomials. In particular we shall find new integral representations, expansion formulae, generating functions, as well as some characterizations of these polynomials.

In a forthcoming paper we shall study the second solution of the differential equation (1.1) and the polynomials associated with the Bessel polynomials.

2. **Preliminaries.** We shall adopt in this paper a somewhat different notation from that used by the previous authors.

First of all we shall consider only the case $b = 2$. As Burchnall points out, we can do this without loss of generality.

Secondly we let $Y_n^{(\alpha)}(x)$ denote the Bessel polynomials $y_n(x, \alpha + 2, 2)$ in the notation of Krall and Frink. But we shall use $y_n(x)$ for the special case $\alpha = 0$.

Sometimes we shall find it convenient to consider the following polynomial [7]

$$\Theta_n^{(\alpha)}(x) = x^n Y_n^{(\alpha)}(1/x). \tag{2.1}$$

In the notation of the hypergeometric series, the Bessel polynomials are given by

$$Y_n^{(\alpha)}(x) = {}_2F_0\left(-n, n + \alpha + 1; -\frac{x}{2}\right) \qquad (n = 0, 1, 2, \cdots) \tag{2.2}$$

Agarwal found the integral representation

$$Y_n^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha + n + 1)} \int_0^{\infty} t^{\alpha+n}\left(1 + \frac{xt}{2}\right)^n \exp(-t)\, dt, \tag{2.3}$$

Received March 4, 1957. The author wishes to express his gratitude to Professor L. Carlitz for his help and guidance in preparing this paper.