A THEOREM OF MYERS

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We prove here a simple extension of the theorem of Myers [2] which says: if the mean curvature of a complete Riemannian manifold M is bounded below by a positive number, then M is compact. (Myers also gets a bound on the diameter of M but we do not.) Our theorem gives the same conclusion, that M is compact, under the weaker assumption (in addition to completeness of M) that the integral of the mean curvature of the tangential direction is infinite along all geodesic rays emanating from some fixed point of M. *M will always denote a complete connected $C\infty$ Riemannian manifold of dimension $d \geq 2$.*

Lemma 1. *If there is a point m in M such that every geodesic ray emanating from m has a point conjugate to m along that ray, then M is compact.*

Proof. We recall the definition of a conjugate point. Let M_m be the tangent space to M at m. Let *exp* be the natural mapping of $M_m \to M$ which carries each ray γ emanating from 0 (in M_m) into the geodesic tangent to γ (at m), and preserving arc length. A point p in M_m is *conjugate* to m if and only if *dexp* is singular at p, i.e. *dexp* has a non-zero kernel at p. If $\sigma = exp\ \gamma$, $n = exp\ p$, then n is *conjugate* to m along σ if and only if p is conjugates to m.

A well known theorem says that geodesics do not minimize arc length beyond conjugate points. Another well known theorem says, under the assumption of completeness, that any two points of M can be joined by a geodesic which does minimize arc length. Together these clearly imply, as Myers and others have pointed out, that if the set F of first conjugate points in M_m is bounded, then M is compact. We now prove F is bounded.

Let S be the unit sphere in M_m. Define the real-valued function f on S by: if $s \in S$, then $f(s)$ = distance from 0 to the first conjugate point on m on the ray in M_m from 0 through s. By assumption, f is defined (and finite) on all of S. A theorem of Morse [1; 235, Lemma 13.1] says f is continuous. Hence f is bounded, so F is bounded, proving Lemma 1.

We consider the following condition on M:

Condition C. There is a point $m \in M$ with the following property. Let γ be any ray in M_m, starting at 0 and running forever; i.e., γ is a curve in M_m defined by $\gamma(u) = up$ for some $p \neq 0$ in M_m. Let $\sigma = exp\ \gamma$. Let $\sigma^*(u)$ be the tangent vector to σ at $\sigma(u)$. Let C be the function defined on $[0, \infty)$ by $C(u)$ = mean curvature of $\sigma^*(u)$. Then our condition is that for every ray (from this

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