NOTE ON BOUNDS FOR CERTAIN DETERMINANTS

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Introduction. The problem of finding bounds for certain determinants, whose exact value would be too complicated to calculate, has concerned mathematicians for many years. Many of the papers which have been written on this subject give positive bounds for the absolute value of determinants with dominant main diagonal, i.e. determinants with elements, a_{ij}, such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad (i = 1, \cdots, n) \tag{1}$$

Among the most recent of these are the papers by J. L. Brenner, [1], A. Ostrowski [5], G. B. Price [7], and Y. K. Wong [8]. Others show that certain determinants which may not have dominant main diagonal, nevertheless do not vanish. For example, I mention the papers of J. L. Brenner [2], and A. Ostrowski [6]. A previous paper by the author [4] also gives results in both these categories.

This paper deals with determinants with non-negative elements which do not necessarily satisfy (1) nor the hypotheses given in the papers mentioned above. In §3 some new criteria are derived for the non-vanishing of determinants, e.g. it is shown that, if 1) the maximum non-diagonal element in each row is not more than k times the minimum element in that row, for any $k \geq 2$, and 2) the diagonal element is at least $(n + 1)$ $[(k - 1)/k]$ times the maximum element, the determinant does not vanish.

Section 4 gives a general criterion which is a sufficient condition that $|A + B| \geq |A| + |B|$ (where $|A|$ means "determinant of A").

Using these results, positive lower bounds, depending on k and n, are found for certain determinants. These bounds are often better than those previously obtained, particularly for large values of n.

To prove the theorems in §3 it is necessary first to quote the following theorem and corollary.

2. A theorem and corollary from a previous paper, [4].

THEOREM A. *Let A be a matrix with positive elements, a_{ij}. If there exist non-negative numbers, c_i, such that the differences,*

$$b_{ij} = a_{ij} - c_i \geq 0 \qquad (i, j = 1, \cdots, n) \tag{2a}$$

have the properties

$$b_{ii}b_{hh} \geq (\sum_{j \neq i} b_{ij})(\sum_{j \neq i} b_{hj}) \qquad (i, h = 1, \cdots, n;\ i \neq h) \tag{2b}$$

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