REMARK ON A PAPER OF O. G. OWENS

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In a recent paper [1] it was shown by an ingenious but rather involved proof that if $u(x, y, t)$ is the solution of the wave equation

$$u_{tt} = u_{xx} + u_{yy} \tag{1}$$

taking the values of a homogeneous polynomial $\Psi(x, y)$ on the characteristic cone $(t^2 - x^2 - y^2 = 0)$, then u is a homogeneous polynomial in (x, y, t) of the same degree as Ψ.

Our purpose is to give a simple proof of this by exhibiting $u(x, y, t)$ explicitly. It is clearly sufficient to solve the problem for the value $\Psi = x^p y^q$.

It is well known [2] that any function of the form

$$(x^2 + y^2 + z^2)^{p+q+\frac{1}{2}} \frac{\partial^{p+q}}{\partial x^p \, \partial y^q} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \tag{2}$$

is a spherical harmonic, that is, a homogeneous polynomial of degree $(p + q)$ satisfying Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{3}$$

It is also well known that if in a solution of (3) analytic in z at $z = 0$ one puts $z = i\,t$, the resulting function of x, y, and t satisfies equation (1). If this is done with the function (2), one obtains, except for a constant factor, the solution

$$(t^2 - x^2 - y^2)^{p+q+\frac{1}{2}} \frac{\partial^{p+q}}{\partial x^p \, \partial y^q} (t^2 - x^2 - y^2)^{-\frac{1}{2}}. \tag{4}$$

One easily sees that this is again a homogeneous polynomial of degree $p + q$, and that when $t^2 - x^2 - y^2 = 0$ it takes the values $[(2p + 2q)!\, 2^{-p-q}/(p + q)!]$ $x^p y^q$. One needs only divide the function (4) by the coefficient of this monomial to obtain the solution of (1) taking the values $x^p y^q$ on the characteristic cone. It is easily seen that (4) is even in t and has the same parity in x and y as $x^p y^q$.

The extension to higher dimensions is obvious.

References

1. O. G. Owens: *Polynomial solutions of the cylindrical wave equation*, this Journal, vol. 23(1956), pp. 371-384.
2. E. W. Hobson: *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge, 1931, p. 131.

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