RADICALS IN FUNCTION RINGS

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1. **Introduction.** This paper is a study of radicals on the class of function rings, or f-rings [2]; the definition and basic properties of f-rings are reviewed in §2 below.

The concept "radical of a ring" is interpreted in the following sense [3]: let C be a category of rings and their homomorphisms and let U be a subclass of rings of C; for each $R \varepsilon C$, let $\rho_U(R)$ be the set of all elements of R which are mapped onto zero by every homomorphism (in C) of R onto a ring of U. Thus ρ_U is a function on the objects of the category C such that $\rho_U(R)$ is a two-sided ideal of R. The functions ρ_U obtained in this way are called radicals on the category C and for each $R \varepsilon C$, the ideal $\rho_U(R)$ is called the ρ_U-radical of R. It follows from the definition that a ring R in C is isomorphic to a subdirect union of rings of the class U if and only if $\rho_U(R)$ is the zero ideal [6]. Thus, the radicals are natural tools for studying the various subdirect decompositions of rings. The object of this paper is to characterize various radicals ρ_U on the category of f-rings in terms of identities satisfied by the elements of $\rho_U(R)$.

2. **Function rings.** A *lattice-ordered ring*, or l-ring [2] is an associative ring which is lattice-ordered such that $x \rightarrow x + a$ is a lattice automorphism for all a, and $x \rightarrow ax$ and $x \rightarrow xa$ are order endomorphisms whenever $a \geq 0$. If, in addition,

(1) $\qquad a \wedge b = 0, x \geq 0 \text{ implies } xa \wedge b = ax \wedge b = 0,$

then the l-ring is called a *function ring*, or f-ring. An l-ring whose ordering is simple is called an ordered ring. Any ordered ring is an f-ring.

The following notation is borrowed from the theory of l-groups [1]: $a^+ = a \vee 0, a^- = a \wedge 0, |a| = a^+ - a^-, R^+ = \{a^+ \mid a \varepsilon R\}$. For any a, $a^+ + a^- = a$. Also, if $a, b, c \geq 0$, it is easy to show that $a \wedge (b + c) \leq (a \wedge b) + (a \wedge c)$. In any l-ring, $|ab| \leq |a||b|$; in an f-ring, the equality prevails.

A homomorphism of an l-ring R will always mean a homomorphism of R considered as a ring and as a lattice. The homomorphic image of an l-ring R is determined to within isomorphism by the coset of zero—the kernel of the homomorphism. This kernel I is always an *l-ideal* of R, that is, I is a two-sided ring ideal of R which satisfies: $a \varepsilon I$, $|b| \leq |a|$ implies $b \varepsilon I$. Conversely, any l-ideal of R determines a homomorphism of R onto the l-ring R/I of cosets of I. The homomorphic image of any f-ring (ordered ring) is again an f-ring (ordered ring).

Definition 1. Let I be an l-ideal of the f-ring R. Then I is called an *l-prime* ideal, or lp-ideal, if $a \not\varepsilon I$ and $b \not\varepsilon I$ together imply $|a| \wedge |b| \not\varepsilon I$.

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