REGIONS OF FLATNESS FOR ANALYTIC FUNCTIONS AND THEIR DERIVATIVES

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1. Introduction. The concept of regions of flatness was introduced by J. M. Whittaker [5] as an infinite set of regions in which the maximum and minimum modulus are in some sense of the same order. After giving a definition for the case of circular regions Mandelbrojt and Ulrich [3] proceeded to state conditions under which the successive derivatives of a function have a given set of regions as regions of flatness. The following development gives corresponding results in the case where the regions are simply connected. Also a restriction that the diameters of the regions be bounded from zero has been eliminated. Thus one may apply the results to study the behavior of a function in the neighborhood of a singular point.

A few theorems are restated in forms convenient for application here. The first is a theorem of Mandelbrojt [2; 188] giving necessary and sufficient conditions that a family of functions be normal. For convenience, a family of functions which has the property that at most a finite set of functions takes the value zero on each compact subset of a domain D will be said to be of non-zero type in D.

THEOREM 1.1. *Let $\mathfrak{F}$ be a family of functions $F(z)$ holomorphic in a domain D. Suppose $\mathfrak{F}$ is of non-zero type in D. Let Δ be a domain such that $\Delta^- \subset D$ (Δ^- denotes the closure of Δ) and form the quantities*

$$m(F) = \max_{z,w\in\Delta^-} \frac{\log \mid F(z) \mid}{\log \mid F(w) \mid}$$

$$m'(F) = \max_{z,w\in\Delta^-} \frac{\mid F(z) \mid}{\mid F(w) \mid}$$

$$L(F) = \min\,[m(F),\, m'(F)]$$

for each function $F(z)$ which is non-zero on Δ^-. A necessary and sufficient condition that $\mathfrak{F}$ be normal in D is that the set of quantities $L(F)$ be bounded for each choice of Δ.

The next theorems from Mandelbrojt's theory of kernels [2] lead to results on the family of derivatives of the functions of a normal family.

THEOREM 1.2. *Let $\{F_n(z)\}$ be a sequence of functions holomorphic in a domain D such that $F_n(z) \rightarrow \infty$ subuniformly* (i.e. *uniformly on each compact subset*) *in D.*

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