THE SOLUTIONS OF A CLASS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER IN A REGION CONTAINING A MULTIPLE TURNING POINT

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1. **Introduction.** The object of this paper is a study of the solutions of the differential equation

$$\frac{d^3w}{dz^3} + h_1(z, \lambda)\frac{d^2w}{dz^2} + \lambda^2\left\{h_2(z, \lambda)\frac{dw}{dz} + h_3(z, \lambda)w\right\} = 0, \tag{1.1}$$

in a region about a so-called *turning point.* λ is a complex parameter that is unbounded in absolute value. The asymptotic forms of the solutions relative to λ are to be determined. The coefficient functions $h_j(z, \lambda)$; $j = 1, 2, 3$, are taken to be expressible as power series in $1/\lambda$ with coefficients that are analytic functions of z.

The characteristic or auxiliary algebraic equation associated with the differential equation (1.1) is

$$\chi^3 + h_2(z, \infty)\chi = 0. \tag{1.2}$$

The roots of this are distinct except at a point which is a zero of $h_2(z, \infty)$, where they all coincide. Such a point of coincidence is called a *turning point*, or *transition point.* Because, in this instance, more than two roots coincide there, we designate the turning point to be *multiple.*

The description of the functional structure of the solutions of a differential equation (1.1), as they depend upon λ, is much more intricate when the z-region considered includes a turning point than when it does not. In the presence of a turning point the region cannot be dealt with as a whole. For near the turning point the forms of the solutions are affected by the fact that the largeness of $|\lambda|$ is, in a way, counteracted by the smallness of the coefficient $h_2(z, \lambda)$. And more remotely from the turning point the z-region must be sub-divided into parts, in each of which separate descriptions of the solutions are requisite. In each of these sub-regions certain solutions have exceptional forms. These must be singled out, and their forms, not only in this sub-region, but in all others too, must be determined.

Differential equations of the second order with turning points are important in quantum mechanics and in other fields of applied mathematics. A considerable, but still incomplete, body of theory for such equations exists. For differential equations of orders higher than the second, existing theory is very fragmentary. For equations of the third order the author has discussed the case

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