ON GENERAL DIRICHLET SERIES

By Donald Clayton Rose

1. **A General Theorem.** In this paper we shall be concerned with the analytic extension of functions defined by general Dirichlet series. The method employed was developed initially by Lindelöf [5] for use in the case of Taylor series. Cowling [3] has extended these results to ordinary Dirichlet series.

Let

$$\mathfrak{f}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

be a general Dirichlet series with finite abscissa of convergence, σ_c. Here the sequence, $\{a_n\}$, consists of real or complex numbers, and $\{\lambda_n\}$ is an unbounded, monotonic, increasing sequence of real numbers. Let D be a simply-connected region in the (u, v) plane bounded by the rectifiable curves whose equations are $v = -\psi(u)$ and $v = \phi(u)$. These functions are defined for one of the two ranges for u, $u \geq A$ or $u \leq A$. We suppose that $\psi(u)$ and $\phi(u)$ are positive and bounded away from zero for $u > A + \epsilon$ or $u < A - \epsilon$ depending upon the range of u for which $\psi(u)$ or $\phi(u)$ are defined and $\epsilon > 0$ is arbitrarily small. We suppose that A is positive and non-integral and a common zero of $\psi(u)$ and $\phi(u)$. D contains the real axis for all w for which $\mathrm{Re}[w] > A$. Suppose there exist functions $a(w)$ and $\lambda(w)$ analytic in and on the boundary of D with the possible exception of the point at infinity such that $a(n) = a_n$ and $\lambda(n) = \lambda_n$ for $n = [A] + 1, [A] + 2, \cdots$, where $[x]$ means the greatest integer contained in x.

Let $\chi_1(w) = (e^{2\pi i w} - 1)^{-1}$ and $\chi_2(w) = (1 - e^{-2\pi i w})^{-1}$. We note the following result. If $w = u + iv$ is bounded away from the integers then $|\chi_1(w)|$ and $|\chi_2(w)|$ are bounded [4; 341], and if K is a suitable constant, then

$$|\chi_1(w)| = |e^{-2\pi i w}|\,|\chi_2(w)| = e^{2\pi v}\,|\chi_2(w)| < Ke^{2\pi v},$$

and

$$|\chi_2(w)| = |e^{2\pi i w}|\,|\chi_1(w)| = e^{-2\pi v}\,|\chi_1(w)| < Ke^{-2\pi v}.$$

We now consider the contour integrals

$$\int_\Gamma \chi_i(w) a(w) e^{-s\lambda(w)}\, dw$$

for $i = 1, 2$ where Γ is the closed path consisting of the three parts Γ_1, Γ_2, and Γ_3. Γ_1 is the path $v = -\psi(u)$ from $w = A$ to $w = \rho_1 C - i\psi(\rho_1 C)$, where C is greater than A and bounded away from the integers in value. $\Gamma_2 = \Gamma_{21} + \Gamma_{22} + \Gamma_{23}$ where Γ_{21} is the (possibly null) line segment $v = -\psi(\rho_1 C)$ from $\rho_1 C - i\psi(\rho_1 C)$ to $C - i\psi(\rho_1 C)$, Γ_{22} is the line segment $u = C$ from $C - i\psi(\rho_1 C)$

Received June 17, 1955. The author wishes to thank Professor V. F. Cowling for help in preparing this paper.