SECTIONS IN PRINCIPAL FIBRE SPACES

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Introduction. Let X be a Hausdorff topological space, and G a group of homeomorphisms of X onto itself. The group G is said to be a *unisolvable, regular transformation group acting on the right of* X if the following conditions are satisfied:

(i) The mapping $(x, g) \rightarrow g(x) = xg$ of $X \times G$ onto X is continuous;

(ii) $(xg_1)g_2 = x(g_1g_2)$, $x \varepsilon X$, $g_1, g_2 \varepsilon G$;

(iii) $x = xg$ if and only if $g = e$, the identity element of G, $x \varepsilon X$;

(iv) Let $D = \{(x, y) : y = xg\} \subset X \times X$, $x, y \varepsilon X$, $g \varepsilon G$. Then the mapping $(x, y) \rightarrow g$ of D onto G is continuous; (the conditions (iii) and (iv) constitute the *unisolvability* condition).

(v) Let B be the space of orbits of G. Then B is a Hausdorff space when given the identification topology. (This is the *regularity* condition for G).

Let p be the natural projection of X onto B. Then p is an open, continuous function [8]. The collection $\mathfrak{B} = \{X, B, p, G\}$ is called a *principal fibre space* [8; 3].

The principal fibre space $\mathfrak{B}$ is said to have a *local cross section* if there is a neighborhood $U \subset B$ and a continuous function f mapping U into X such that $pf(b) = b$ for $b \varepsilon U$. If $U = B$, then $\mathfrak{B}$ is said to have a *(full) cross section.* A necessary and sufficient condition that a principal fibre space be a principal fibre bundle is that it have a local cross section defined for each $b \varepsilon B$ and some U_b containing b. (A proof of this last statement is given below for the sake of completeness. Although a proof is not contained in the literature, it is undoubtedly well-known). A necessary and sufficient condition that X be homeomorphic to $B \times G$ is that $\mathfrak{B}$ have a full cross section.

It has long been known that if G is a Lie group acting on an analytic manifold, then local cross sections exist [4; 109–110]. In 1950, Gleason [7] proved that if G is a compact Lie group, and X is completely regular, then local cross sections exist. With only slight alterations, his proof can be extended to the case where G is a locally compact (i.e., arbitrary) Lie group. (Again, for the sake of completeness, and since a proof of this well-known and important result is not contained in the literature, a proof is given below). In the same year, using Gleason's (generalized) result, Serre [18] and Borel [3] proved that if G is a locally compact group, and B is locally contractible, locally compact, and paracompact, then local cross sections exist. Recently the author [14] demonstrated the existence of local cross sections when X is a separable, metric, locally compact group of finite dimension, and G is a closed subgroup of X.

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