A THEOREM ON UPPER SEMI-CONTINUOUS DECOMPOSITIONS

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1. **Introduction.** If $\mathcal{G}$ is an upper semi-continuous collection of compact sets filling up a topological space M, $\mathcal{G}$ is called an *upper semi-continuous decomposition* of M. If we topologize $\mathcal{G}$ by defining an open set as any subcollection $\mathcal{U}$ such that the union of all sets of $\mathcal{G}$ belonging to $\mathcal{U}$ is open (in M), we obtain what is known as the *decomposition space*, denoted by $\mathfrak{D} = \mathfrak{D}(M, \mathcal{G})$. It is the principal object of this paper to prove the following theorem, which gives conditions sufficient for the homeomorphism of $\mathfrak{D}$ and M.

Theorem. *Let M be a complete, locally compact, separable, metric space and $\mathcal{G}$ a topologically contracting collection which is an upper semi-continuous decomposition of M. Suppose that given any non-degenerate member C of $\mathcal{G}$, any open set U containing C, and any $\epsilon > 0$, there exists a point $p \in C$ and a mapping g of M onto itself such that* (i) $g(C) = p$, (ii) *g maps $M - C$ homeomorphically onto $M - p$,* (iii) *g is stationary outside of U, and* (iv) $\delta[g(D)] < \delta(D) + \epsilon$ *for each $D \in \mathcal{G}$, where $\delta(S)$ = diameter of S. Then M and $\mathfrak{D}(M, \mathcal{G})$ are homeomorphic.*

A somewhat similar result was obtained by Wardwell [2] for compact metric spaces, but it is easy to give examples showing that the two results are independent. Under Wardwell's hypotheses the collection of non-degenerate members of $\mathcal{G}$ cannot be dense in M, but on the other hand $\mathcal{G}$ need not be topologically contracting.

Let $E_n (n \geq 1)$ denote Euclidean n-space and suppose $\mathcal{G}$ is an upper semi-continuous decomposition of E_n into compact convex sets. Anderson and Klee [1] conjectured that E_n and $\mathfrak{D}(E_n, \mathcal{G})$ are homeomorphic. We are unable to prove this, but if the non-degenerate sets of $\mathcal{G}$ are required to be strictly convex (defined in §5), the conclusion follows from the theorem of this paper (see §5, Corollary 2).

2. **Definitions and notation.** We refer the reader to [3; Chap. VII] for general information about upper semi-continuous collections and decomposition spaces.

A collection $\mathcal{G}$ of sets in a topological space is said to be *topologically contracting* if, given any infinite sequence $\{C_n\}$ of distinct members of $\mathcal{G}$ and sequences $\{x_n\}$ and $\{y_n\}$, where x_n and y_n are points of $C_n (n = 1, 2, \ldots)$, either $\{x_n\}$ and $\{y_n\}$ converge to the same point or else neither converges. In a metric space it is easy to show that a collection $\mathcal{G}$ is topologically contracting if and only if given any infinite sequence $\{C_n\}$ of distinct members of $\mathcal{G}$ for which lim inf $\{C_n\}$ is non-vacuous, then $\lim_{n\to\infty} \delta(C_n) = 0$.

If $\mathcal{G}$ is any collection of sets, we denote by $\mathcal{G}^*$ the set of all points belonging

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