CHARACTERIZATIONS OF FOURIER-STIELTJES TRANSFORMS

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1. Introduction. Given a locally compact Abelian group G with character group $\tilde{G}$, we say that a complex-valued function $\varphi(x)$ on G satisfies the condition (B) if for some constant M

$$\text{(B)} \qquad \left| \sum a_n \varphi(x_n) \right| \leq M \sup_{\tilde{x} \in \tilde{G}} \left| \sum a_n (x_n , \tilde{x}) \right|$$

for all finite sets of complex numbers (a_n) and points (x_n) in G. If there exists a bounded Radon measure $\tilde{\mu}$ on $\tilde{G}$ such that

$$\text{(F-S)} \qquad \varphi(x) = \int_{\tilde{G}} (x, \tilde{x}) \, d\tilde{\mu}(\tilde{x}),$$

φ is called a Fourier-Stieltjes transform.
A straightforward calculation shows that a Fourier-Stieltjes transform (F-S) satisfies the condition (B) with $M = \| \tilde{\mu} \| =$ total variation of $\tilde{\mu}$. Our main result is the converse:

Theorem 1. *If φ is measurable and satisfies* (B) *there exists a (unique) Radon measure $\tilde{\mu}$ on $\tilde{G}$ such that* (F-S) *holds nearly everywhere in x and $\| \tilde{\mu} \| \leq M_0$, where M_0 is the smallest value of M satisfying* (B). *If φ is continuous, then* (F-S) *holds everywhere and $\| \tilde{\mu} \| = M_0$.*

This result for the additive group of the reals (R_1) is due to Bochner [1] in the continuous case. The generalization to measurable φ on R_1 by a quite different method is due to Phillips [3]. It turns out that the extension to general groups of Bochner's methods in proper conjunction with a criterion of Schoenberg [4] yields the representation for measurable φ almost automatically. In this respect, and since we rely essentially on standard results from the theory of harmonic analysis on groups, the present treatment has some claim to definitiveness.

2. The Schoenberg criterion. Given a function f in $L^1(G)$ denote as usual its Fourier transform $F(f)$ by $\tilde{f}$, where

$$\tilde{f}(\tilde{x}) = \int_G (x, \tilde{x}) \, dx,$$

and recall that the mapping $F: f \to \tilde{f}$ is a $1 - 1$ norm decreasing linear map of $L^1(G)$ onto a dense subset of $C_0(\tilde{G})$, where $C_0(\tilde{G})$ denotes the Banach space

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