THE ENCLOSING OF SIMPLE ARCS AND CURVES BY POLYHEDRA

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1. **Introduction.** It is a purpose of this paper to formulate a local condition on the imbedding of an arc or curve in three space that eliminates certain pathological difficulties known to exist in general. This condition is necessary (but not sufficient) for the tame imbedding of such a set. It is shown that if an arc has *property* $\mathcal{P}$ the complementary set is an open 3-cell in the compactified 3-space. It is not at all clear that if a simple closed curve has property $\mathcal{P}$ and is unknotted in some sense, for instance, unknotted in terms of the knot group, that the complement of the curve is homeomorphic to the complement of an ordinary circle. A proper local-global condition to insure this latter situation will be given later in an attack on the problem of determining the class of tamely imbedded arcs and curves.

To motivate the definition of property $\mathcal{P}$ consider an arc J in a plane π. (Such an arc is equivalent to an interval by known accessibility and extension theorems.) Suppose p is a point of J and ϵ a positive number. There is a simple closed curve K in π containing p in its interior of diameter less than ϵ that meets J in one or two points according as p is or is not an end-point of J. Even more can be said, K may be taken as "locally polyhedral" (see §3) except where it meets J. For an arc J in 3-space, R, we replace "curve K" by "simple closed surface K" and adopt this condition (which can no longer be proved) as the defining property of the class of arcs and curves under consideration.

The basic result, due to J. W. Alexander, that a polyhedral 2-manifold of genus zero in a compactified Euclidean 3-space is the common boundary of two closed 3-cells is used continually [1]. The concept of a semi-linear map, used extensively by Graeub [3] and Moise [5], is also basic.

2. **Notations.** If J is a subset of R and ϵ a positive number, $S(J, \epsilon)$ is the ϵ-neighborhood of J. The null set is denoted by $\square$. The word *polygon* will be an abbreviation for simple, closed, polyhedral image of a circumference. *Surface* will mean a topological 2-sphere and *disk* a closed, topological 2-cell.

3. **Definitions.** The set K is called *locally polyhedral* at p if there is a neighborhood of p in R meeting K in a finite (or null) polyhedron [4]. The set J is said to have property $\mathcal{P}$ provided that to each $x \in J$ and each $\epsilon > 0$ there is a set $K(\epsilon, x)$ such that (i) $K(\epsilon, x)$ is a topological 2-sphere, (ii) $x \in$ interior of $K =$ *int* K, (iii) diameter of K is less than ϵ, (iv) cardinal $K \cap J =$ order (Menger-Urysohn) of x in J, and (v) K is locally polyhedral at each point of $K - J$ (the set of points in K, but not in J).

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