AN ALGEBRA DETERMINED BY A BINARY CUBIC FORM

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1. **Introduction.** The device of writing a given quadratic form as the square of a linear form was used by Clifford to define what is now known as a Clifford algebra and was used much later by Dirac to obtain the equations of the electron spin. The success of this theory in the quadratic case suggests an investigation of the algebra similarly defined in terms of an algebraic form of degree greater than two. This paper is concerned with the case of the binary cubic form.

Let $A \equiv A(a, b, c, d)$ denote the linear associative algebra generated by quantities R and S over a coefficient field F where R and S are required to satisfy

$$(1.1) \qquad (Rx + Sy)^3 \equiv ax^3 + bx^2y + cxy^2 + dy^3$$

for all values of the variables x and y in F, all elements of which are assumed to commute with R and S. The coefficients a, b, c, and d are fixed elements in F (which is required to have more than two elements) and the unit element of A is identified with that of F so that the right member of (1.1) is in A.

The main results are listed below. They are independent of the choice of a, b, c, and d. (1) A is an infinite dimensional algebra over F and an extensive class of bases is identified. (2) Every properly homomorphic image of A is finite dimensional which in turn implies that A is not a direct sum. (3) In case the cubic form in question has nonproportional linear factors the center Z of A is isomorphic to $F[t_1, t_2]/(t_1^3 + t_2^2 + k^2t_2)$ where $F[t_1, t_2]$ denotes the polynomial domain in two commuting indeterminates t_1 and t_2 over F and $(t_1^3 + t_2^2 + k^2t_2)$ is the indicated principal ideal ($k \neq 0$). In the remaining cases (assuming $3 \neq 0$) Z is isomorphic to $F[t_1, t_2]/(t_1^3 + t_2^2)$. Here F is assumed extended, if necessary, so that the binary cubic can be put in a standard form. (4) Z contains a polynomial domain $F[t]$. A has a basis of 18 elements over $F[t]$ as operator domain.

2. **Defining relations and definitions.** Having assumed F to have at least three elements, condition (1.1) is equivalent to the following set of relations.

$$(2.1) \qquad R^3 \equiv a.$$

$$(2.2) \qquad S^3 \equiv d.$$

Received August 16, 1952; in revised form, April 14, 1954; presented to the American Mathematical Society, September 6, 1951. This paper is a condensation of a Ph.D. dissertation written at the University of Tennessee under the direction of Professor Wallace Givens. The author wishes to thank Professor Givens for his assistance.