ADDITIVE POLYNOMIALS

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Introduction. Results. By an additive polynomial (a.p.), or p-polynomial, over a field k we mean one with coefficients in that field satisfying the identity

$$f(x + y) = f(x) + f(y), \tag{1}$$

where x and y are independent indeterminates. If the characteristic of k is zero the a.p.'s are trivial, but if it is a prime p they are the finite sums

$$f(x) = \sum \alpha_\nu x^{p^\nu} \tag{2}$$

with ν running over nonnegative integers. They have been studied by Ore [4], [5], Hasse and Witt [2], and Artin [1].

The basic principles of a.p. are so completely developed in [4], [5], [6] that one would not expect new results without introducing further restrictions on k. We were led to such restrictions by some questions about generalized local class field theory. These we will discuss in another paper; it is enough to mention here that, if we call a subgroup $\mathfrak{v}$ of the additive group k^+ of a field k *open* when there is a polynomial $u(x)$, over k, such that $\mathfrak{v} \supset u(k) = \{u(\alpha) \mid \alpha \in k\}$, then the study of norm groups of higher ramified extensions of regular local fields [9] involves certain open subgroups of the residue class field. (See, for example, [9], Proposition 7, noting that the assumption of algebraic closure is not used in its proof.) When k is a Galois field the open subgroups are simply all subgroups but in other cases they are very interesting as the following summary of our results shows.

Let k be any field satisfying

Axiom 1. *k has characteristic $p \neq 0$.*

Axiom 2. *k has no inseparable extension.*

Axiom 3. *k has for each integer n at most one extension* (*in any algebraic closure*) *of degree n and has exactly one extension of degree p.*

Then a subgroup of k^+ is open if and only if it is the set $f(k^+)$ of values of some a.p. $f(x)$; for each such $f(x)$, $(k^+ : f(k^+))$ equals the number of zeros of $f(x)$ in k (hence is finite, and a power of p); for every open subgroup $\mathfrak{v}$ there is an a.p. $g(x)$ such that $\mathfrak{v} = g(k^+)$, $g(x)$ has all its zeros in k, and every a.p. $f(x)$ with $f(k^+) \subset \mathfrak{v}$ is of form $g(h(x))$. The additive group k^+, considered as vector space over its prime subfield, has a linear topology [3; 74] in which "open subgroup" has the meaning defined above; the a.p.'s induce an everywhere dense subring of the ring of all continuous endomorphisms of k^+, under a natural topology on that ring.

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