SINGULAR QUARTIC FORMS

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1. **Introduction.** We are interested in characterizing singular forms over finite fields. By a singular zero of a form (*i.e.*, a homogeneous polynomial) we mean a common zero of the form and all of its formal partial derivatives. A singular form is a form for which all zeros lying in the field of coefficients of the form are singular zeros. Such a characterization is of value in the consideration of algebraic surfaces over number fields as well as over finite fields. Several recent papers have demonstrated the usefulness of a complete description of forms over $\mathfrak{p}$-adic fields [4], [6] and of forms over function fields with non-zero characteristic [5], which have only singular zeros modulo the prime ideal $\mathfrak{p}$. This latter problem is essentially equivalent to characterizing singular forms over finite fields.

It is easy to give a complete description of singular forms over finite fields of degree two and three. Through an application of a theorem of Carlitz [2] we are able to do likewise for singular quartic forms over sufficiently large finite fields.

2. **An extension of a theorem of Carlitz.** We shall let k denote a finite field of characteristic p with $q = p^n$ elements. k^e will denote the multiplicative group of the e-th powers of the non-zero elements of k. $\mathfrak{V}^{(t)}$ will be a t-dimensional vector space over k. $E_1 = (1, 0, \cdots, 0)$, $E_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $E_t = (0, 0, \cdots, 0, 1)$ will be a basis of $\mathfrak{V}^{(t)}$.

We restate Carlitz's theorem [2]:

THEOREM. *Let $f(x)$ be a polynomial of degree m in $k[x]$. Suppose e is a factor of $q - 1$ and that $f\colon k \to k^e \cup 0$. Then $f(x) = g^e(x)$, provided $q \geq \lambda(m)$; where $\lambda(m)$ is a monotonic increasing function and $\lambda(m) \geq m$.*

Carlitz first gave a proof of this theorem [1] for the case $e = 2$. In this proof he made use of Weil's theorem on the Riemann hypothesis in function fields and he showed that $(m - 1)^2 \geq \lambda(m)$. Later he gave a proof [2] (for all e) which made use of more elementary results, but led to a large bound for $\lambda(m)$. Actually in studying Carlitz's proof it can be seen that he proved a somewhat stronger theorem, *i.e.*, the condition that $f\colon k \to k^e \cup 0$ may be replaced by the property that $f(a) \in k^e$ for at least $q - r$ elements a of k, provided we have $q \geq \lambda(m, r)$; λ monotonic increasing in both variables.

For our purposes, we need an extension of the above theorem. (The stronger version would also carry over.)

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