THE IMBEDDING OF A RING AS AN IDEAL IN ANOTHER RING

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If R is a ring with unit element and if R is an ideal of a ring S, then $S = R \oplus R'$ where R' is the annihilator of R in S. Thus the ring R is an ideal of the ring S if and only if there exists a unique ring R' such that $S = R \oplus R'$. However, if the given ring R does not have a unit element, then the exact nature of a ring S containing R as an ideal is not so apparent. Some special cases of the general problem of determining all rings S containing a given ring R as an ideal are discussed in the first section of this paper.

The second section is concerned with the relationship between the ideal structures of a given ring R and any ring S containing R as an ideal. The principal theorem of this section exhibits a homomorphism of a set of weakly prime ideals of S onto the set of weakly prime ideals of R. Special instances of this result give the relationship between the prime ideal structures as well as the regular ideal structures of R and S.

1. **Imbedding theorems.** A ring R will be called *right faithful* if $Ra = 0$, $a \varepsilon R$, implies $a = 0$. Our conclusions that follow on right faithful rings obviously will hold also for left faithful rings.

The additive group R^+ of a ring R is made into an (R, R)-module in the usual way. Thus if we let a' be the element of R^+ corresponding to the element a of R, we have $ab' = a'b = (ab)'$ for all $a, b \varepsilon R$.

Considering R^+ as a left R-module, let $E(R)$ denote the ring of all endomorphisms of R^+. Hence each $c \varepsilon E(R)$ is an endomorphism of R^+ such that $a(b'c) = (ab')c$ for all $a, b \varepsilon R$. Evidently R may be considered as a subring of $E(R)$ in case R is a right faithful ring.

1.1 Lemma. *The right faithful ring R is a right ideal of $E(R)$.*

Proof. For each $a \varepsilon R$ and $c \varepsilon E(R)$, there exists $b \varepsilon R$ such that $a'c = b'$. Since $x'(ac) = (xa')c = xb' = x'b$ for every $x' \varepsilon R^+$, we have $ac = b$. This proves 1.1.

A corollary of this lemma is that $a'c = (ac)'$ for each $a \varepsilon R$, $c \varepsilon E(R)$. Consequently, if $Rc = 0$, $c \varepsilon E(R)$, then $R^+c = 0$ and $c = 0$. In other words, the right annihilator of R in $E(R)$ is zero.

Let us designate the normalizer of R in $E(R)$ by $N(R)$. We recall that $N(R)$ is the largest subring of $E(R)$ in which R is an ideal.

1.2 Theorem. *If the ring R is an ideal of the ring S and if the right annihilator of R in S is zero, then S is isomorphic to a subring of $N(R)$.*

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