HOMOGENEOUS SPACES

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1. **Introduction.** This paper summarizes and refines results of Kuratowski [2] and van Dantzig [3] on homogeneous spaces in general. It provides a counterexample to a query of van Dantzig [3] and a few lemmas which may help toward solution of the problem he raised: does there exist a noninvolutory homogeneous group? Now every group is bihomogeneous, and every Abelian group is involutory homogeneous. The example given is a noninvolutory homogeneous manifold made up of uncountably many planes. Further, it is shown that every microhomogeneous connected linearly ordered space is locally Birkhoff homogeneous.

We list below six properties of a topological space, together with the defining relations. These properties have been previously studied in [2] and [3].

(a) *Homogeneity*: for two points, x, y, there is an automorphism (homeomorphism of space onto itself) sending x to y.

(b) *Bihomogeneity*: there is an automorphism sending x to y and y to x.

(c) *Involutory homogeneity*: there is an involution sending x to y (hence y to x).

(d) *Microhomogeneity*: there are neighborhoods U of x, V of y, and a topological equivalence $\varphi : U \rightarrow V$, $\varphi(x) = y$.

(e) *Almost homogeneity*: there is a homeomorphism *into* sending x to y.

(f) *Two-point homogeneity*: for any x, y, distinct, and z, w, distinct, there is an automorphism sending x to z and y to w.

The concepts (a)-(e) suggest point relations, which are equivalence relations only in cases (a) and (d). We abbreviate both "homogeneous" and "equivalent" with the initial h. The letters b, i, m, a, will be used similarly. The context will make it clear which meaning is to be taken for the ambiguous abbreviations. We refer to the h-equivalence classes (transitivity sets) as *rooms*, and to the m-equivalence classes as *m-rooms*.

We shall call a linearly ordered space *Birkhoff homogeneous* if it is order-isomorphic to all its open intervals; G. D. Birkhoff's linear homogeneous continua are obtained by the further requirement that all bounded monotone simple sequences converge. We add the following two concepts:

1.1. Definition. Two points are *semi-equivalent* if they have neighborhood bases which may be put into one-one correspondence so that corresponding terms are homeomorphic.

The property does not imply m-equivalence; it is an equivalence relation and determines in the obvious way s-rooms and s spaces.

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