MOMENT SPACES AND INEQUALITIES

By Melvin Dresher

1. Introduction. An integral inequality can be interpreted geometrically as a condition that a prescribed point lie in a space determined by the convex hull of a given curve. Once the boundary of the convex hull is characterized, we can obtain integral inequalities by requiring that points in a space lie within the boundaries of the space.

We use this general procedure to show how the inequalities of Hölder, Minkowski and Jensen can be obtained, and then derive a rather general inequality. All of these inequalities have the common geometric property that they follow from the same set of boundaries, the so-called lower boundaries of the convex hull. Use of the upper boundaries of the space yields new inequalities. In general, it is analytically very difficult to characterize the upper boundaries. We give some special examples of new inequalities obtained by using the upper boundary of the convex hull.

2. Generalized moment space. We summarize some properties of moment spaces. Let $\Phi(x)$ be a distribution function over the interval [0, 1] and let $f_1(x), f_2(x), \cdots, f_m(x)$ be a set of m continuous functions. The *i-th moment* of Φ is defined by

$$r_i(\Phi) = \int_0^1 f_i(x)\, d\Phi(x).$$

We define the *m-th moment space, R^m*, as the set of points $r = (r_1, r_2, \cdots, r_m)$ in m-dimensional Euclidean space whose coordinates are the moments $r_1(\Phi)$, $r_2(\Phi), \cdots, r_m(\Phi)$ of at least one distribution function, Φ.

We have shown [1] that R^m is closed, bounded and convex. Further, we showed that if C^m is the curve traced out in m dimensions by

$$r_i = f_i(x) \qquad (i = 1, 2, \cdots, m)$$

as x varies between 0 and 1, then R^m is identical with the convex set spanned by C^m. It follows that the extreme points of R^m, those points of R^m in which a plane and a convex set intersect in exactly one point, are on C^m,

3. Boundary of moment space. A point $\rho = (\rho_1, \rho_2, \cdots, \rho_m)$ of R^m is said to lie on the boundary of R^m if and only if ρ is in R^m and there exists a set of real numbers $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i r_i + \alpha_0 \geq 0$$

Received September 3, 1952; in revised form, February 6, 1953. Presented to the American Mathematical Society, Dec. 27, 1951.