SOME SUMS ANALOGOUS TO DEDEKIND SUMS

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1. **Introduction.** The Dedekind sum $s(h, k)$ is defined by means of [6], [7]

$$s(h, k) = \sum_{r \pmod{k}} \left(\left(\frac{h}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{1.1}$$

where $((x)) = x - [x] - \frac{1}{2}$ when x is not an integer, $((x)) = 0$ when x is an integer. The most interesting property of the sum is the reciprocity formula

$$12hk\{s(h, k) + s(k, h)\} = h^2 + 3hk + k^2 + 1. \tag{1.2}$$

Elsewhere [5] the writer has used the representation

$$s(h, k) = \frac{1}{4k} + \frac{1}{k} \sum_{\zeta \neq 1} \frac{1}{(\zeta^{-1} - 1)(\zeta^h - 1)}, \tag{1.3}$$

where ζ runs through the k-th roots of unity, to give a simple proof of (1.2) and indeed of Apostol's generalization [1].

The object of the present paper is to discuss some analogs of (1.1) in the field $GF(q, x)$. The straightforward analog of (1.1) is of little interest in the present case. Instead we consider an analog suggested by the representation (1.3). We are thus led to a set of reciprocity theorems similar to (1.2) as well as to a number of related formulas that may be of interest in themselves. The discussion makes use of certain functions previously defined by the writer [2], [3]; the relevant properties are reproduced below.

2. **Notations and preliminaries.** By $GF(q, x)$ we shall understand as usual the field of rational functions of the indeterminate x with coefficients in $GF(q)$. By $\Phi = GF\{q, x\}$ is meant the field consisting of the numbers

$$\alpha = \sum_{r=-\infty}^{m} c_r x^r \qquad (c_r \in GF(q)), \tag{2.1}$$

where m is an arbitrary integer (positive, negative or 0); if $c_m \neq 0$ we define $\deg \alpha = m$, $|\alpha| = q^m$, so that $|\alpha|$ is a valuation for Φ. For our purpose we shall also require the extension $\Phi' = \Phi(z)$, where $z^{q-1} = x$.

We next define [2], [3]

$$\psi(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^r}}{F_r} \qquad (t \in \Phi'), \tag{2.2}$$

where

$$F_r = \prod_{s=0}^{r-1} (x^{q^r} - x^{q^s}), \qquad F_0 = 1;$$

Received August 9, 1952.