CONTINUOUS AND EQUICONTINUOUS COLLECTIONS OF ARCS

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A collection G of point sets is said to be *upper semi-continuous* (*continuous*) provided that if h is the sequential limiting set of a sequence of elements of G and g is an element of G which intersects h, then g contains (is) h. A collection G of arcs is said to be *equicontinuous* provided that for every positive number p, there exists a positive number q such that if x and y are two points of an arc g of G at a distance apart less than q, then the diameter of the interval xy of g is less than p.

The purpose of this paper is to prove the following theorem: *If, in the plane, G is a continuous and equicontinuous collection of mutually exclusive arcs and their sum is closed and compact, then there exists a reversibly continuous transformation of the plane into itself which carries each arc of G into a straight line interval.*

The notation $d(x,y)$ will denote the distance between the points x and y and $\mathrm{cl}(M)$ the point set M plus its boundary. If G is a collection of point sets, G^* will denote their sum. The notation $\{g_n\}$ denotes the sequence $g_1, g_2, g_3, \cdots$.

Theorem 1. *Suppose G is a continuous and equicontinuous collection of mutually exclusive arcs in a metric space and $\{g_n\}$ is a sequence of arcs of G converging to the arc g of G. Then if A and B are the end points of g, the end points of g_n can be labeled A_n and B_n in such a manner that $\{A_n\} \to A$ and $\{B_n\} \to B$.*

Theorem 2. *If G is a continuous and equicontinuous collection of mutually exclusive arcs in a metric space and G^* is a compact continuum and K is the set of all end points of the arcs of G, then K is closed and it is not the sum of three mutually separated point sets.*

With the aid of Theorem 1, it is easy to show that K is closed and that the supposition that it is the sum of three mutually separated point sets leads to a contradiction.

Theorem 3. *If G is a continuous collection of mutually exclusive arcs in the plane and K is the set of all end points of the arcs of G, then every continuum lying in K is a continuous curve.*

Lemma 3.1. *If M is a continuum in the plane which is not connected im kleinen, there exist two mutually exclusive simple closed curves J_1 and J_2, a connected domain D bounded by $J_1 + J_2$, a sequence $\{A_n\}$ of points of J_1 converging to a*

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