REPRESENTATION OF CERTAIN BANACH ALGEBRAS ON HILBERT SPACE

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Introduction. Denote by $\mathfrak{X}$ a real Banach space and by $\mathfrak{B}$ the algebra of all bounded linear operators on $\mathfrak{X}$. Assume given a mapping $T \to T^*$ of $\mathfrak{B}$ into itself with the following properties: (1) $(T^*)^* = T$, (2) $(T_1 + T_2)^* = T_1^* + T_2^*$, (3) $(T_1T_2)^* = T_2^*T_1^*$, and (4) $TT^* = 0$ implies $T = 0$. Under these assumptions, Kakutani and Mackey [7; Theorem 4] have shown that a real inner product (x, y) can be introduced into $\mathfrak{X}$ such that the norm $| x | = (x, x)^{\frac{1}{2}}$ is equivalent to the given norm in $\mathfrak{X}$ and $(xT, y) = (x, yT^*)$ for all $x, y \in \mathfrak{X}$ and $T \in \mathfrak{B}$. They also obtain an analogous result for the complex case provided $\mathfrak{X}$ is infinite dimensional [8; Theorem 4].

The main purpose of the present note is to extend the above results to general Banach algebras with minimal ideals. More precisely, let $\mathfrak{A}$ denote a Banach algebra which contains a minimal right ideal R and in which there is defined a mapping $a \to a^*$ satisfying conditions (1)–(4) above. The problem is to introduce into R an inner product (x, y) such that $(xa, y) = (x, ya^*)$ for $x, y \in R$ and $a \in \mathfrak{A}$. Provided R satisfies certain dimension restrictions, this can always be done. In general, however, additional conditions are needed for the norm $| x | = (x, x)^{\frac{1}{2}}$ to be equivalent to the given norm in R. The dimension restrictions also enable us to drop condition (2) on the mapping $a \to a^*$. Our results also supplement results obtained by Kaplansky [11; Theorem 7.3].

In §§1, 2 below are collected some preparatory lemmas concerning arbitrary rings. In §3 the main theorems for general Banach algebras are proved. In §4 the general results are applied to the case of dense [4; 229] algebras of linear operators on a linear vector space to obtain more immediate generalizations of the Kakutani-Mackey results.

1. **A mapping theorem.** This section is devoted to the proof of a lemma which is an extension of a lemma proved elsewhere by us [14; 760]. First consider an arbitrary ring $\mathfrak{R}$ which contains a minimal right ideal R such that $R\mathfrak{R} \neq (0)$. For each $a \in \mathfrak{R}$, there is associated an endomorphism T_a of the additive group in R defined by $xT_a = xa$, $x \in R$. Since R is minimal, the class of all such T_a is an irreducible ring of endomorphisms of R. Therefore the class $\mathfrak{D}$ of all $\mathfrak{R}$-endomorphisms of R (that is, endomorphisms which commute with each T_a) is a division ring. If R is considered as a left linear vector space $\mathfrak{R}$ over $\mathfrak{D}$, then the T_a constitute a dense ring of linear transformations on $\mathfrak{R}$ [4; Theorem 6]. In other words, if $x_1, \cdots, x_k, y_1, \cdots, y_k$ (k arbitrary) are

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