q-BERNOULLI NUMBERS AND POLYNOMIALS

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1. **Introduction.** We define a set of "numbers" η_m by means of the symbolic formula

$$(q\eta + 1)^m = \eta_m \qquad (m > 1), \qquad \eta_0 = 1, \qquad \eta_1 = 0;$$

and a set of polynomials $\eta_m(x) = \eta_m(x, q)$ in q^x such that

$$\eta_m(x + 1) - \eta_m(x) = mq^x[x]^{m-1}, \qquad \eta_m(0) = \eta_m ,$$

where $[x] = (q^x - 1)/(q - 1)$. We next define a set of numbers β_m by means of $\beta_m = \eta_m + (q - 1)\eta_{m+1}$ and a set of polynomials $\beta_m(x) = \beta_m(x, q)$ such that

$$q^x\beta_m(x) = \eta_m(x) + (q - 1)\eta_{m+1}(x), \qquad \beta_m(0) = \beta_m .$$

Some properties of the η's and β's are discussed in §§4, 5. For $q = 1$, β_m reduces to the Bernoulli number B_m ; η_m however does not remain finite.

By means of the numbers $a_{m,s}$ defined in §3 (which generalize the Stirling numbers of the second kind) we arrive at certain explicit expressions for β_m . And finally using these expressions we derive the main result of the paper—a partial generalization of the Staudt-Clausen theorem. We have

$$\beta_m = \sum_{k=2}^{m+1} N_{m,k}(q)/F_k(q),$$

where $F_k(q)$ denotes the cyclotomic polynomial and $N_{m,k}(q)$ is a polynomial in q which satisfies

$$(q - 1)^{m-1}N_{m,k}(q) \equiv qF'_k(q) \sum_{1 \le sk \le m+1} (-1)^{m+1+sk}\binom{m}{sk - 1} \qquad (\text{mod } F_k(q)).$$

For additional properties of β_m see §7.

In conclusion (§8) we define numbers ϵ_m such that $\epsilon_0 = 1$,

$$q(q\epsilon + 1)^m + \epsilon^m = 0 \qquad (m > 0),$$

and polynomials $\epsilon_m(x) = \epsilon_m(x, q)$ such that

$$q\epsilon_m(x + 1) + \epsilon_m(x) = [2]\,[x]^m, \qquad \epsilon_m(0) = \epsilon_m .$$

The product

$$2^{-m}(q + 1)^m(q^4 + 1)(q^6 + 1) \cdots (q^{2m+2} + 1)\epsilon_m(\tfrac{1}{2}, q^2)$$

may be considered a q-generalization of the Euler numbers.

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