THE ZEROS OF THE DERIVATIVES OF THE REAL ELLIPTIC ϑ_3 -FUNCTION

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1. Let

$$\theta_q(x) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos nx, \tag{1}$$

for real x and for

$$0 < q < 1, \tag{2}$$

denote the elliptic theta-function ϑ_3 (with the unit of the independent variable changed so that the period is 2π, rather than 1). It has recently been shown [2] that, for every q on the interval (2), the first and second derivatives of the positive function (1) have exactly two (simple) zeros on any period, say $-\pi < x \leq \pi$. The object of this paper is to investigate the zeros of the higher order derivatives of $\theta_q(x)$.

Let $\theta_q^m(x)$ denote the m-th derivative of (1); so that,

$$\theta_q^{2m}(x) = 2(-1)^m \sum_{n=1}^{\infty} n^{2m} q^{n^2} \cos nx \qquad (m = 1, 2, \cdots), \tag{3_m}$$

and

$$\theta_q^{2m+1}(x) = 2(-1)^m \sum_{n=1}^{\infty} n^{2m+1} q^{n^2} \sin nx \qquad (m = 0, 1, \cdots). \tag{4_m}$$

Since $\theta_q^m(x)$ is a periodic function, its number $N_m(q)$ of zeros (counting multiplicities) on a period $-\pi < x \leq \pi$ is even. Between every two zeros of $\theta_q^m(x)$, there is at least one zero of $\theta_q^{m+1}(x)$ by Rolle's theorem, so that $N_{m+1}(q) \geq N_m(q) - 1$. But since $N_m(q)$ is even for every m,

$$N_{m+1}(q) \geq N_m(q) \qquad (0 < q < 1). \tag{5}$$

It follows (see [1]) from the general theories of either Sturm or Laguerre that $N_m(q)$ is a non-decreasing step-function of q on the interval (2). It will be shown that, if $m > 0$, then:

(i_m) $N_{2m-1}(q) = N_{2m}(q)$ for $0 < q < 1$;

(ii) $N_{2m}(q)$ attains exactly m values, namely, $2, 4, \cdots, 2m$;

(iii) $N_{2m}(q)$ is continuous from the right (and, if $m > 1$, its discontinuities separate and are separated by those of $N_{2m-2}(q)$);

(iv_m) the power series

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