THE IDENTICAL VANISHING OF THE LAPLACE INTEGRAL

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The following theorem is contained in a recent paper by the present writer. (See [3]. The present form of the theorem follows easily from a theorem of Blaschke through a simple geometrical consideration as set forth in the addendum to the paper cited.)

Let $f(z)$ be a function analytic in the interior of the unit circle $C : | z | = 1$, and satisfying the condition

$$| f(z) | < \frac{M}{(1 - | z |)^\sigma} \qquad (| z | < 1),$$

where $M > 0$, $\sigma > 0$, are constants. Then

(A) *$f(z)$ vanishes identically if it vanishes on a sequence of points z_n, $| z_n | < 1$, on the real axis, or, more generally, on a circle orthogonal to C, such that*

$$\sum_{n=1}^{\infty} (1 - | z_n |) = \infty;$$

(B) *$f(z)$ vanishes identically if it vanishes on a sequence of points z_n, $| z_n | < 1$, interior to the circle*

$$\Gamma : | z - \tfrac{1}{2} | = \tfrac{1}{2},$$

such that

$$\sum_{n=1}^{\infty} (1 - | 2z_n - 1 |) = \infty.$$

As an application, we propose, in this note, to consider the identical vanishing of the Laplace integral

$$f(s) = \int_0^{+\infty} e^{-st} F(t)\, dt, \tag{1}$$

$F(t)$ being an L-function according to the definition of Doetsch (see [1; 13]; we use Rs to denote the real part of s); thereby a more general form is obtained for the following two known theorems:

Theorem 1. (Lerch [2; 345–348]) *Let s_0 be a point at which* (1) *converges. If $f(s)$ vanishes on a sequence of points s_n :*

$$s_n = s_0 + nd \qquad (d > 0; n = 1, 2, \cdots),$$

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