THE DETERMINATION OF CONNECTED LINEAR SECTIONS

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In 1913 Brunn [2] established the following theorem about sets in an n-dimensional Euclidean space $\mathfrak{R}_n$, $n \geq 2$.

Theorem (Brunn). *Let x be a point in $\mathfrak{R}_n$, $n \geq 2$, which has the property that each line through x intersects a bounded closed set S in a non-empty connected set. The Kerneigebiet is defined to be the set of all such points x having the above property.*

Then the Kerneigebiet is a closed, convex set which is contained in S.

One can say that the set S in the above theorem is *star-like* [1; 3] relative to each point of the Kerneigebiet. Hence S is a continuum when the Kerneigebiet is not null. In the following theory the set S is to be a continuum (a compact, connected set) in $\mathfrak{R}_n$, $n \geq 2$. A hyperplane L is the $(n - 1)$-dimensional set of points $x \in \mathfrak{R}_n$ satisfying a linear equation $f(x) = c$, and the intersection $L \cdot S$ determined by L is called a linear section of S. In order to generalize the concepts developed by Brunn, the following theorem provides our point of departure.

Theorem 1. *Consider a property P on hyperplanes in $\mathfrak{R}_n$, $n \geq 2$. Let x be a point in $\mathfrak{R}_n$ such that each hyperplane through x has property P, and designate the set of all such points x by K.*

Then each component of K is a convex set.

Proof. Let x_1 and x_2 be any two points in a component C of K, and designate the straight line segment joining x_1 and x_2 by l_{12}. Choose *any* point $r \in l_{12}$, and let L be *any* hyperplane passing through r. If $l_{12} \subset L$, then $C \cdot L \neq 0$. If $l_{12} \not\subset L$, then x_1 and x_2 are on opposite sides of L. Since C is *connected*, we must have $L \cdot C \neq 0$. Hence in all cases $K \cdot L \neq 0$, so that L must have property P. Since r was any point on l_{12}, and since L was any hyperplane through r, $l_{12} \subset C$. Consequently C is a convex set, and Theorem 1 has been proved.

Remark. It should be observed that Theorem 1 also holds in a linear space.

In all of the following theorems the set K has the following definition.

Definition 1. Suppose S is a continuum in $\mathfrak{R}_n$, $n \geq 2$. Let x be a point in $\mathfrak{R}_n$ such that each hyperplane L through x intersects S in a connected set, and designate the set of all such points x by K. (A linear section $L \cdot S$ may or may not be empty.)

This definition differs from that for the Kerneigebiet in that *empty* intersections are admissible. Also hyperplanes replace the rôle played by straight lines.

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