AN IMBEDDING THEOREM FOR PARACOMPACT METRIC SPACES

By C. H. Dowker

J. Dieudonné [1] has investigated a class of spaces which he calls paracompact. The paracompact spaces include the compact spaces and also include the metric separable spaces. In this paper we show that each paracompact metric space can be topologically imbedded in a generalized (not necessarily separable) Hilbert space (also called an abstract Euclidean space or a unitary Banach space), and that each subset of a generalized Hilbert space is paracompact. Of course, the spaces which can be topologically imbedded in a separable Hilbert space are the separable metrizable spaces. Thus, for metric spaces, paracompactness may be regarded as an immediate generalization of separability.

A covering of a space R by open sets (hereafter all coverings will be understood to be coverings by open sets) is called *locally finite* if every point x of R has a neighborhood meeting only a finite number of the open sets of the covering. (We follow the terminology of Dieudonné [1]. Lefschetz [2; 13] calls locally finite coverings "neighborhood-finite" and uses "locally finite" for a more restricted class of coverings.) A covering $\mathfrak{V}$ of a space R is called a refinement of a covering $\mathfrak{U}$ of R if every set V of $\mathfrak{V}$ is contained in some set U of $\mathfrak{U}$. A space R is called paracompact if every covering of R has a locally finite refinement.

A generalized real Hilbert space H (see the discussion of complex Hilbert spaces [4]) can be defined in terms of coordinates as follows. Let $\Omega : \{\alpha\}$ be a set of arbitrary potency. Let a set $x : \{x_\alpha\}$ of real numbers x_α, one for each α of Ω, be a point of $H(= H(\Omega))$ whenever $\sum x_\alpha^2$ is finite. Thus, for each point x, at most a countable number of the coordinates x_α are different from zero. The distance $\rho(x, y)$ between two points x and y of H is defined to be $(\sum (x_\alpha - y_\alpha)^2)^{\frac{1}{2}}$. This distance function is defined for all pairs of points, and, with this distance, H is a metric space. In particular, if Ω is countably infinite, H is a separable Hilbert space, and, if Ω is finite, H is a Cartesian space.

Lemma 1. *Every paracompact metric space is homeomorphic to a subset of a generalized Hilbert space.*

Proof. Let R be a paracompact metric space. For each positive integer r, let $\mathfrak{V}_r$ be the covering of R by all spherical neighborhoods of radius $(2r)^{-1}$. Let $\mathfrak{U}_r : \{U_{r\alpha}\}$ be a locally finite refinement of $\mathfrak{V}_r$. Let Ω be the totality of sets $U_{r\alpha}$ for all integers r. Let H be the Hilbert space $H(\Omega)$. A point of H is a set $x : \{x_{r\alpha}\}$ of real numbers $x_{r\alpha}$, one corresponding to each $U_{r\alpha}$, such that $\sum_{r\alpha} x_{r\alpha}^2$ is finite.

A transformation of R into H is defined as follows. If $p \in R$, let $\phi_{r\alpha}(p)$ be the distance from p to the complement of $U_{r\alpha}$. Let

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