THE LOCATION OF THE CRITICAL POINTS OF SIMPLY AND DOUBLY PERIODIC FUNCTIONS

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By the use of conformal transformations, especially those involving the exponential function, known theorems concerning the location of the zeros of the derivative of a polynomial or other rational function and the critical points of a harmonic function are here employed to prove new results on the location of the zeros of the derivative of a periodic analytic function and the critical points of a periodic harmonic function.

Conformal transformations have been used on numerous previous occasions in the study of critical points; a geometry defined in one plane can be transformed onto another plane, whether the original geometry is euclidean (compare [1]) or non-euclidean (compare [2]). Results in the new plane may be expressed alternately in terms of the original geometry or of the geometry after transformation.

1. **Simply periodic functions.** Let the analytic function $f(w)$ have the period $2\pi i$:

$$f(w + 2\pi i) \equiv f(w), \tag{1}$$

where $w = u + iv$. We introduce the frequently used convention of artificial "end-points" of the period strip $v_0 < v \leq v_0 + 2\pi$, and assume that $f(w)$ has no singularities other than poles in each strip, end-points included. The left-hand end-point is a zero, a non-zero point of analyticity, or a pole according as we have

$$\lim_{u \to -\infty} \frac{| f(u + iv) |}{e^{nu}} = A \neq 0,$$

where n is positive, zero, or negative. The right-hand end-point is a zero, non-zero point of analyticity, or a pole, according as we have

$$\lim_{u \to +\infty} \frac{| f(u + iv) |}{e^{nu}} = A \neq 0,$$

where n is negative, zero, or positive. In each case $| n |$ is necessarily integral and is the order of the pole or zero. The order of the function $f(w)$ is the sum of the orders of its poles in a period strip, end-points included, which is equal to the sum of the orders of its zeros in a period strip, end-points included.

We proceed to establish

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