EXTENDING A METRIC

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In this paper we shall prove a covering theorem (Theorem 1) and apply it to get three metrization theorems (Theorems 2, 3, 5).

A topological space is metric provided that for each point P and each point Q, there exists a number $d(P, Q)$ such that

(a) $d(P, Q) = 0$ if and only if $P = Q$,

(b) $d(P, Q) = d(Q, P)$ (symmetry),

(c) $d(P, Q) \leq d(P, R) + d(R, Q)$ for each point R (triangle condition), and

(d) the function $d(x, y)$ preserves limit points.

By (d) we mean that P is a limit point of the set M if and only if for each positive number ϵ, there is a point Q of $M - M \cdot P$ such that $d(P, Q)$ is less than ϵ. The number $d(P, Q)$ is called the distance between P and Q. The distance function $d(x, y)$ of a space is called the metric of the space.

The distance between two points is a positive number. This may be seen by letting P equal Q in (c) and applying (a) and (b). Lindenbaum [7] has shown that we may omit (b) and substitute

(c′) $d(P, Q) \leq d(P, R) + d(Q, R)$

for (c). Then (b) and (c) follow from (a) and (c′). Birkhoff [3; 466] uses the inequality

(c″) $d(P, Q) \leq d(Q, R) + d(R, P)$

in place of (c′) and shows that (b) follows from (a) and (c″).

A covering theorem. The theorem in this section may be useful in studying a topological space some of whose properties are defined by means of a sequence of collections of sets covering the space. A space satisfying R. L. Moore's Axiom 1 [8] is such a space. In particular, we shall show that the theorem may be used in solving three metrization problems.

A collection G of point sets is *coherent* provided that each proper subcollection G' of G contains an element which intersects an element of $G - G'$.

THEOREM 1. *Suppose that r is a positive integer and $H_1, H_2, \cdots$ are collections of sets such that each pair of points that can be covered by a coherent collection of r or fewer elements of H_{i+1} can be covered by an element of H_i. If P and Q are two points whose sum cannot be covered by any element of H_s but which can be covered by a coherent collection of sets $h_1, h_2, \cdots$, and h_n belonging to $H_{\alpha(1)}, H_{\alpha(2)}, \cdots$, and $H_{\alpha(n)}$ respectively, then*

$$(1) \qquad 2(1/r^{\alpha(1)} + 1/r^{\alpha(2)} + \cdots + 1/r^{\alpha(n)}) > 1/r^s .$$

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