A NOTE ON THE NUMBER THEORY OF QUATERNIONS

By Ivan Niven

1. We prove the following result.

Theorem. *Every Hurwitz integral quaternion is expressible as a sum of three squares of (Hurwitz) integral quaternions. A Lipschitz integral quaternion is expressible as a sum of three squares of (Lipschitz) integral quaternions if and only if it has the form*

$$a + 2bi + 2cj + 2dk.$$

Rational integers are denoted by italic type.

These results are best possible; for example, we shall show that $6 + 2i$ is not a sum of two squares of integral quaternions in either the Hurwitz or Lipschitz sense.

2. **The Lipschitz case.** We prove that any quaternion $a + 2bi + 2cj + 2dij$, which we shall write as $(a, 2b, 2c, 2d)$, is expressible as a sum of three squares of quaternions (x, y, z, w). This can be done by observing that

$$(a, 2b, 2c, 2d) = (1, b, c, d)^2 + (u, 0, 0, 0)^2 + (0, r, s, t)^2$$

is equivalent to the equation

$$a = 1 - b^2 - c^2 - d^2 + u^2 - r^2 - s^2 - t^2.$$

Integers u, r, s, t can be chosen to satisfy this equation for any values a, b, c, d.

Conversely it is clear that (a, b, c, d) is not a sum of squares if any one of b, c, d is odd.

3. **The Hurwitz case.** Since the Hurwitz integral domain contains the Lipschitz integral domain, we need merely treat integers of the forms

$$\left(\frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}, \frac{a_4}{2}\right)$$

with a_1, a_2, a_3, a_4 odd, and (a_1, a_2, a_3, a_4) with at least one of a_2, a_3, a_4 odd.

The first of these cases can be handled in the manner of the last section, that is, by the equation

$$\left(\frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}, \frac{a_4}{2}\right) = \left(\frac{1}{2}, \frac{a_2}{2}, \frac{a_3}{2}, \frac{a_4}{2}\right)^2 + (u, 0, 0, 0)^2 + (0, r, s, t)^2,$$

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