SINGULAR PERTURBATIONS OF NON-LINEAR OSCILLATIONS

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The perturbation method is one of the strongest tools for establishing periodic solutions of systems of ordinary differential equations. Suppose a system of n equations

$$X'_\mu = F_\mu(X_1, \cdots, X_n; \epsilon) \qquad (\mu = 1, \cdots, n)$$

for n functions $X_\mu(t)$ with $X'_\mu = dX_\mu/dt$ is given whose right-hand members depend on a parameter ϵ, and do not involve the independent variable t directly. Suppose that for $\epsilon = 0$ the equations possess a known periodic solution, the "base solution". With the aid of well-known perturbation methods one can then obtain periodic solutions of the equations for sufficiently small values of ϵ in the neighborhood of the base solution, provided the right members of the differential equations depend regularly on ϵ, for $\epsilon = 0$. The first, and decisive step in justifying the perturbation procedure is the theorem of Poincaré that, under appropriate conditions and for sufficiently small values of ϵ, a unique periodic solution exists which, as ϵ approaches zero, tends to the base solution for $\epsilon = 0$. (See [1; 401], [5], [6], [7; Chapters II–IV], [8]).

There exists, however, an important class of perturbation problems which is not covered by Poincaré's investigations. In Poincaré's theory essential use is made of the assumption that the functions $F_\mu(X_1, \cdots, X_n; \epsilon)$ are continuous functions of ϵ in the neighborhood of $\epsilon = 0$. (Poincaré even limits his discussion to functions F_μ which are regular analytic in all variables.) This implies, in particular, that the order of the differential system is the same for $\epsilon = 0$ as for $\epsilon \neq 0$. We shall term such problems *regular*. While such regular perturbation problems are common in the field of non-linear vibrations, (see [3], [5], [6], [8] and literature mentioned there), there exists also an important type of perturbation problems in which the dependence on the parameter ϵ is singular in such a way that the order of the system of differential equations is reduced for $\epsilon = 0$. Usually, such systems have the form of equations (1) below. Problems of this nature we shall call *singular* perturbation problems. They arise in mechanical problems concerned with the oscillations of elastically bound masses, if one of the masses is much smaller than the others; for oscillations in electrical networks, if the inductivity or the capacity of one of the meshes is very small. (It seems, though, that these problems have not been given the attention they deserve. But see [4] where such cases of non-linear electrical oscillations are considered in detail.)

As we shall show, it remains true for such singular problems that, under appropriate conditions and for sufficiently small values of ϵ a unique periodic

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