EXPANSION OF FUNCTIONS IN COMBINATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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The purpose of this paper is to consider the expansion of a suitable arbitrary function of a real variable in a series of solutions of a linear differential equation containing a parameter. These solutions may be expressed in terms of the generalized hypergeometric function. The interval on which the function is to be defined contains a regular singular point of the differential equation, either as an end or interior point. G. D. Birkhoff [1], M. H. Stone [5] and [6], J. D. Tamarkin [7], and many others have considered expansions of functions in solutions of differential equations but only in [6] does the interval contain a singularity of the differential equation. The differential equation considered here is self-adjoint and of a simple type with a regular singular point at the origin. There are either two-point boundary conditions with the origin separating the points, or one-point boundary conditions with certain regularizing conditions at the origin. The methods are those used by Stone in [5] and [6]. As special cases there result Fourier series, Fourier-Bessel and Dini expansions in Bessel functions. Also new expansions in Bessel functions may be obtained.

I. The Origin as an Interior Point

1. **Setting of the problem.** Consider the linear differential equation of order n

$$L(y) + \lambda y \equiv L(y) + \rho^n y = 0, \tag{1}$$

where

$$L(y) \equiv y^{(n)} + * + a_2 x^{-2} y^{(n-2)} + \cdots + a_n x^{-n} y. \tag{2}$$

The $a_2, \cdots, a_n$ are real or complex constants determined so that $L(y)$ is a self-adjoint expression. λ and ρ are parameters. The adjoint equation of (1) may be written

$$\pm L(v) + \lambda v = 0 \qquad (+ \text{ if } n \text{ is even}, - \text{ if } n \text{ is odd}). \tag{3}$$

If r is a root of the indicial equation of (1) then $-r + n - 1$ is also a root. Assume n to be either of the form 2μ or $2\mu + 1$, μ an integer. Then if r_k, $k = 1, \cdots, n$, are the roots we can write $r_k = \nu_k + (n-1)/2$ where ν_k, $k = 1, 2, \cdots, \mu$, are real or complex numbers and $\nu_k = -\nu_{n-k+1}$, $k > n/2$. The ν_k are assumed arranged so that

$$R(\nu_1) \geq R(\nu_2) \geq \cdots \geq R(\nu_\mu) \qquad (R(\nu_i) = \text{real part of } \nu_i). \tag{4}$$

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