A CERTAIN CLASS OF MULTIPLICATIVE FUNCTIONS

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1. **Introduction.** The function $\psi(n)$, which is defined for every positive integer n, is said to be a multiplicative function if it possesses the property:

$$\text{(A)} \qquad \psi(mn) = \psi(m)\psi(n) \quad \text{when} \quad (m, n) = 1.$$

Most of the multiplicative functions with which I shall be concerned possess one of the further two properties:

$$\text{(B)} \qquad \psi(p^{l+1}) = \psi(p)\psi(p^l) - p^k\psi(p^{l-1}),$$

or

$$\text{(C)} \qquad \psi(p^{l+1}) = \psi(p)\psi(p^l) - (-1)^{\frac{1}{2}(p-1)}\psi(p^{l-1}),$$

where p is an odd prime, k is a fixed non-negative integer, and l is any positive integer.

I write $\mathfrak{M}_k$ for the class of all functions satisfying (A) and (B), and $\mathfrak{M}'$ for the class of all functions satisfying (A) and (C). It will be observed that no properties of the forms (B) or (C) have been stated for powers of the prime number 2. This is due to the exceptional place occupied by the prime 2 in the development of the theory. It will, however, be found that many of the multiplicative functions which arise in the investigations satisfy relations of the form

$$\text{(D)} \qquad \psi(2^{l+1}) = A\psi(2)\psi(2^l) \qquad (l > 0),$$

where A is a fixed number independent of l.

The following general theorem on multiplicative functions is almost trivial:

Theorem 1.1. *If $\psi(n)$ belongs to the class $\mathfrak{M}_k$ (or $\mathfrak{M}'$), then so does the function $f(n)$ defined by*

$$f(n) = \psi(n) \ (n \text{ odd}), \qquad f(n) = B\psi(n) \ (n \text{ even}).$$

It follows as an immediate corollary from the theorem that a function $\psi(n)$ of $\mathfrak{M}_k$ or $\mathfrak{M}'$ possessing property (D) for a non-zero A may be replaced by a multiplicative function of the same class satisfying (D) with $A = 1$. (Take $B = A$ in the theorem.)

Functions of the classes $\mathfrak{M}_k$ and $\mathfrak{M}'$ arise chiefly in the theory of the elliptic modular functions as coefficients in the Fourier expansions of modular forms of negative dimensions, and their multiplicative properties are then most easily and naturally demonstrated by making use of the known principles of that theory. It was by such means that Mordell [24] first proved the multiplicative properties of Ramanujan's function $\tau(n)$.

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