OVERCONVERGENCE, DEGREE OF CONVERGENCE, AND ZEROS OF SEQUENCES OF ANALYTIC FUNCTIONS

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In 1904 Porter published the first known example of a Taylor development of which a suitably chosen sequence of partial sums converges in a region containing both points interior and exterior to the circle of convergence. Porter mentioned that this phenomenon has close relations with the accumulation of zeros of partial sums at points of the circle of convergence. In the intervening years, this general topic has been further and extensively studied by Jentzsch, Ostrowski, Szegö, Pólya, Carlson, Bourion, and a number of others; Bourion [1] has recently published a summary of the results obtained, to which the reader may refer for further historical and technical details.

It is the object of the present paper to undertake an analogous study for sequences of *arbitrary* analytic functions, when we assume neither that the functions are polynomials nor that the sequences are defined in terms of a linear series, of the form

$$\sum_{n=0}^{\infty} a_n q_n(z),$$

where the a_n are constants and where the asymptotic properties of the $q_n(z)$ are known.

Although the results of the present study are of considerable generality, involve geometric and analytic configurations which are extremely broad in scope, and yet include all known results on Taylor's series except those of special form which seem to apply only to sequences of polynomials or to sequences of otherwise heavily restricted functions, their inception is not due merely to a search for greater generality. The present results are the outgrowth of, and have immediate application to, the study of maximal convergence of sequences of polynomials, maximal convergence of sequences of rational functions, convergence of analytic functions of prescribed norm of best approximation, convergence of functions of given measure of approximation and minimum norm, and convergence of interpolating functions of minimum norm. The writer hopes on another occasion to consider further application of the present methods, notably to series of harmonic functions and to Dirichlet series.

The methods employed in the present paper involve especially the continued use of a harmonic majorant, a method employed systematically both by Ostrowski and Bourion; we introduce, however, the concept of *exact* harmonic

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