TWO THEOREMS ON SCHLICHT FUNCTIONS

By Bernard Friedman

In a recent paper [4] R. Salem proved the following:

Let the function $f(z)$ be analytic and schlicht inside the unit circle $|z| < 1$. Let its expansion in the neighborhood of the origin be

$$f(z) = a_{-1}z^{-1} + \sum_0^\infty a_n z^n.$$

If there exists an index p such that for $n \geq p$ all coefficients a_n are rational integers or integers of an imaginary quadratic field, then $f(z)$ is rational.

D. C. Spencer mentioned this result in a seminar on schlicht functions held at New York University and wondered whether it were possible to prove this theorem by elementary means. It turned out that this was possible for part of Salem's theorem. The following theorem is proved:

Theorem 1. *Let*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

be a function regular and schlicht in the unit circle, that is, $f(z_1) = f(z_2)$; $|z_1|$, $|z_2| < 1$ *implies* $z_1 = z_2$. *Let* $a_2, a_3, \cdots$, *the coefficients in the power series expansion of* $f(z)$, *be positive or negative integers or zero; then* $f(z)$ *is one of the following nine functions*:

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad z\,\frac{1 \pm z}{1 \pm z^3}. \tag{2}$$

The second theorem is a slight contribution to the Bieberbach hypothesis concerning a_4. Bieberbach conjectures that for any function schlicht in the unit circle, there exists the inequality

$$|a_n| \leq n \qquad (n = 2, 3, \cdots).$$

The case $|a_2| \leq 2$ is classical [1]. Löwner [2], in 1926, succeeded in showing that $|a_3| \leq 3$. In 1944, Spencer and Shaefer [5] gave another proof that $|a_3| \leq 3$. In both cases the difficulties which prevent the extension of this method to a_4 or higher have not been resolved.

We prove by elementary means

Theorem 2. *For any function schlicht in the unit circle,*

$$|a_4| \leq 4.16.$$

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