A NOTE ON A THEOREM OF TURÁN

By Yun Kai Lu

1. The following theorem is due to Turán [1]:

Let $f(x)$ *be a non-negative, convex upward and odd function in* $(0, \pi)$, *and let*

$$f(x) \sim b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots . \tag{1}$$

We denote the n-th Cesàro mean of order k of the series (1) *by* $S_n^{(k)}(x)$, *then*

$$f(x) \geq S_n^{(1)}(x) > S_n^{(2)}(x) > \cdots > 0 \qquad (0 < x < \pi).$$

This theorem states that, when x and n are fixed, $S_n^{(k)}(x)$ is a non-increasing function of k where k is an integer. The question rises whether the theorem still holds when k is not an integer. (This problem was suggested to the author by Professor K. K. Chen. She is indebted to Professor F. T. Wang for several suggestions.) In this note we give an example showing that the answer is negative.

2. Let

$$f(x) = \begin{cases} 0 & (x = 0), \\ (\pi - x)/2 & (0 < x \leq \pi), \end{cases}$$

and $f(x) = -f(-x) = f(x + 2\pi)$ elsewhere, then $f(x) \sim \sum_{n=1}^{\infty} (\sin nx)/n$ satisfies the conditions of Turán's theorem. If we denote by $\sigma_n^{(\alpha)}(x)$ the n-th Cesàro mean of order α of the Fourier series of $f(x)$, then

$$\sigma_n^{(\alpha)}(x) = \frac{1}{A_n^{(\alpha)}} \sum_{\nu=1}^{n} A_{n-\nu}^{(\alpha)} \frac{\sin \nu x}{\nu} = \sum_{\nu=1}^{n} \frac{a_\nu}{P_\nu(\alpha)} \frac{\sin \nu x}{\nu},$$

where

$$A_n^{(\alpha)} = \binom{n + \alpha}{n} = \frac{(n + \alpha)(n + \alpha - 1) \cdots (\alpha + 1)}{n!},$$

$$a_\nu = \frac{n!}{(n - \nu)!}, \qquad P_\nu(\alpha) = (n + \alpha) \cdots (n - \nu + \alpha + 1).$$

Thus we have

$$\frac{\partial \sigma_n^{(\alpha)}(x)}{\partial \alpha} = - \sum_{\nu=1}^{n} \frac{a_\nu P_\nu'(\alpha)}{P_\nu^2(\alpha)} \frac{\sin \nu x}{\nu} = \sin x \, E_n(x, \alpha),$$

say.

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