A SKEW QUARTIC ASSOCIATED WITH A TETRAHEDRON

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1. **Historical note.** The study of a problem in astronomy led Jean Hachette (1769–1834) to consider the locus of the line of intersection u of two variable orthogonal planes au, $a'u$ revolving about two fixed coplanar axes a, a' [4; 179–180, 311–312]. He thus came upon the orthogonal cone, or the Hachette cone, as the French writers call it.

By taking for the two axes a, a' two skew lines, one of Hachette's associates in the École Polytechnique obtained the orthogonal hyperboloid of one sheet [1; 71]. Steiner pointed out that the circular sections of an orthogonal hyperboloid are perpendicular to the axes a, a' [7; 292, article 65].

The first systematic discussion of both the orthogonal cone and the orthogonal hyperboloid is due to Cha les [2]. An extensive treatment of the subject was given by Schroeter [6; articles 12, 25, 26].

2. **A pencil of quadrics.** (a) Consider a tetrahedron $(T) = ABCD$, $BC = a$, $CA = b$, $AB = c$, $DA = a'$, $DB = b'$, $DC = c'$. We associate with (T) the three orthogonal hyperboloids (aa'), (bb'), (cc') whose respective axes are the pairs of edges a, a'; b, b'; c, c'.

(b) Let M be a point common to the two surfaces (aa'), (bb'). Since M lies on (aa'), the two planes MBC, MAD are perpendicular and therefore the plane through MA perpendicular to MBC passes through MD. Again since M lies on (bb') the plane through MB perpendicular to MCA passes through MD. Consequently [3; 27, article 67] the plane through MC perpendicular to the plane MAB passes through MD, i.e., the two planes MCD, MAB are perpendicular; hence the point M also belongs to the hyperboloid (cc').

Thus any point M common to two of the three surfaces (aa'), (bb'), (cc') also belongs to the third surface, hence: *the three orthogonal hyperboloids associated with a tetrahedron belong to the same pencil of quadrics.*

3. **A skew quartic.**

(a) Definition. The three orthogonal hyperboloids associated with a tetrahedron $(T) = ABCD$ (article 2a) have in common a skew quartic (C_4). This curve will be referred to as *the skew quartic* (C_4) *of the tetrahedron* (T), or *associated with* (T).

(b) Each of the three hyperboloids associated with (T) contains a pair of opposite edges of (T) and therefore contains the vertices of (T), hence: *the skew quartic* (C_4) *of a tetrahedron passes through the vertices of that tetrahedron.*

The skew quartic (C_4) of a tetrahedron (T) is

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