GÂTEAUX DIFFERENTIABILITY AND ESSENTIAL BOUNDEDNESS

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The following observations are meant to unify, by suitable generalization, three theorems which characterize Gâteaux-differentiable functions on complex Banach spaces as Fréchet differentiable by means of topological conditions.

The first of these is due to A. E. Taylor [4], the characteristic property being continuity. The second has been found by Hille [1]; continuity is replaced by local boundedness. The author [5] has used essential continuity; that is, continuity of the partial function which is obtained by dropping a suitable set of the first category from the domain of definition.

Each of these characterizations leads to a theorem which asserts that the limit of certain sequences of Fréchet differentiable functions is Fréchet differentiable. The first theorem corresponds to uniform convergence on compact sets (which incidentally is equivalent to continuous convergence). The second corresponds to what Hille designates as locally bounded convergence. The third requires only what is necessary to make the limit function possess a Gâteaux differential, since the limit of essentially continuous functions is essentially continuous. Uniform convergence on discs of the form $\{x + \zeta h\}$, $|\zeta| \leq \rho(x, h)$ is sufficient.

The first theorem is of course contained in the other two. The relation between the latter is not immediately obvious. In §1 we discuss it topologically, in §3 we apply power series.

1. We define first a property of subsets of a linear vector space X which has been used successfully by Mazur and Orlicz in [2], [3].

(1.1) A subset P of a linear vector space X over the field Φ is an M-O-set, if the space is not the union $\bigcup_1^\infty P_i$ of a sequence of sets $P_i = \alpha_i P + a_i$, where $\alpha_i \in \Phi$, $a_i \in X$.

We note that this is a purely algebraic definition. If X is a Banach space, a set of the first category is an M-O-set; if X is the set of real numbers considered as a linear space over itself in the natural way the sets of measure zero are likewise M-O-sets; since there exist sets of measure zero of the second category an M-O-set is not necessarily of the first category.

A subset of an M-O-set is an M-O-set, but nothing is maintained about finite or countable sums of M-O-sets. A homothetic image $\alpha P + a$ of an M-O-set is an M-O-set.

Now let X be a linear space over a topological field Φ; we define:

(1.2) A subset D of X is linearly open if for $x, h \in X$ the elements ζ of Φ for which $x + \zeta h \in D$ form an open set of Φ.

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