MAXIMAL FIELDS WITH VALUATIONS, II

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In this paper we continue the study of the structure of maximal fields. The problem may be set forth as follows: if K is a maximal field with value group Γ and residue class field $\mathfrak{K}$, to what extent is K determined by Γ and $\mathfrak{K}$? In particular, if K and $\mathfrak{K}$ have the same characteristic, is K necessarily a power series field? For the case where Γ is a discrete group of finite rank, Schilling [5] answers the latter question in the affirmative. In this paper we carry over the affirmative answer to (suitably defined) discrete groups of infinite rank. This result, together with the results and counter-examples given in [1], closes the gap between the necessary and sufficient conditions on Γ, which will now read as follows: characteristic ∞ — no assumption necessary; characteristic p — either Γ discrete or $\Gamma = p\Gamma$. As for $\mathfrak{K}$, the situation of [1] stands unchanged: in the case $\Gamma = p\Gamma$, there remains a slight gap between the necessary condition ($\mathfrak{K}^p = \mathfrak{K}$) and the sufficient condition (every polynomial equation with exponents powers of p has a root in $\mathfrak{K}$).

1. **Preliminary lemmas.** We shall use the notation and definitions of [1] and, in particular, the notion of *pseudo-convergence* will be further exploited. In our first lemma we prove the uniqueness of the maximal extension under a hypothesis unlike those of [1].

Lemma 1. *Let K be a field with a valuation V, and suppose that the pseudo-convergent sets in K which lack a limit in K all have zero breadth. Then the immediate maximal extension of K is uniquely determined, up to analytic isomorphism.*

Proof. Let L be an immediate maximal extension of K. Then L can be obtained from K by a sequence of adjunctions of limits of pseudo-convergent sets [1; Theorems 2 and 3] and it will suffice to prove the uniqueness of a single adjunction. Suppose then we have reached a field M, $K \subseteq M \subset L$. Any element z in L but not in M, is *a fortiori* not in K. By [1; Theorem 1], z is a limit of a pseudo-convergent set $\{a_\rho\}$ in K, without a limit in K. Now let $M(u)$ be another extension of M in which u is also a limit of $\{a_\rho\}$. For any polynomial f we form a Taylor series

$$(1) \qquad f(z) = f(a_\rho) + f_1(a_\rho)(z - a_\rho) + \cdots + f_n(a_\rho)(z - a_\rho)^n,$$

where $f_n(z) = f^{(n)}(z)/n!$ formally. Our hypothesis that $\{a_\rho\}$ has zero breadth means that $V(z - a_\rho)$ becomes arbitrarily large as ρ increases. From (1) it follows that $Vf(a_\rho)$ is either ultimately constant, in which case $Vf(z) = Vf(a_\rho)$; or else $Vf(a_\rho)$ also becomes arbitrarily large, in which case $Vf(z) = \infty$. The same

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