INEQUALITIES FOR FOURIER TRANSFORMS OF POSITIVE FUNCTIONS

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1. Let $f(x)$ be the Fourier transform of a non-negative integrable function $\varphi(t)$, so that

$$f(x) = \int_{-\infty}^{\infty} e^{ixt}\varphi(t)\,dt. \tag{1.1}$$

Then we evidently have $|f(x)| \leq f(0)$ for all real x. If in addition $f(x)$ vanishes for $|x| \geq 1$, we are going to show that

$$|f(\tfrac{1}{2})| \leq \tfrac{1}{2}f(0), \tag{1.2}$$

more generally that

$$|f(1/n)| \leq f(0)\cos \pi/(n+1) \qquad (n = 2, 3, \cdots), \tag{1.3}$$

and still more generally that

$$|f(x)| \leq f(0)\cos \frac{\pi}{[1/x]+1} \tag{1.4}$$

(where $[t]$ denotes the greatest integer not exceeding t).

These inequalities are best possible. For (1.2) we can see this immediately, since $\varphi(t) = 2\pi^{-1}t^{-2}\sin^2 \frac{1}{2}t$ gives

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| > 1, \end{cases}$$

and there is equality in (1.2). It is more difficult to exhibit examples for which equality occurs in (1.3) and (1.4); this will be done in §4.

These inequalities are (as we discovered after proving them) analogues of inequalities for the coefficients of positive trigonometric polynomials which have been established by Fejér [4], Egerváry and Szász [3], and others. In the form in which we have stated them, our results do not include the analogous inequalities for trigonometric polynomials, but it is easy to put them into a more general form of which the inequalities for trigonometric polynomials are special cases (see Theorems 3 and 4).

In §2 we collect a number of known lemmas on determinants; in §3 we prove (1.4) by what seems to be the most natural method; in §4 we show that our inequalities are best possible.

The construction used in §4 was inspired by a formal attack on (1.4) suggested

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