CONDITIONS OF APPLICABILITY OF THE STRONG LAW OF LARGE NUMBERS

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Let a sequence of random variables

$$(1) \qquad x_1, x_2, \cdots, x_n, \cdots$$

be given, possessing finite mathematical expectations, with the distribution functions

$$F_1(x), F_2(x), \cdots, F_n(x), \cdots.$$

Without restricting the generality we may suppose that the mathematical expectations of all random variables (1) are equal to zero, i.e., that

$$Ex_n = \int_{-\infty}^{+\infty} x\, dF_n(x) = 0 \qquad (n = 1, 2, \cdots).$$

Let us introduce the notations

$$b_n = Ex_n^2 = \int_{-\infty}^{+\infty} x^2\, dF_n(x), \qquad b_n(A) = \int_{-A}^{+A} x^2\, dF_n(x),$$

$$s_n = x_1 + x_2 + \cdots + x_n, \qquad v_n = \frac{s_n}{n}.$$

It is said that the sequence (1) obeys the strong law of large numbers, if

$$(2) \qquad P\{\lim_{n\to\infty} v_n = 0\} = 1,$$

where the symbol $P\{\cdots\}$ denotes the probability of the expression enclosed in the braces. So far no conditions have been found, which are simultaneously necessary as well as sufficient for the sequence (1) to obey the strong law of large numbers. In the case of the sequence (1) of mutually independent random variables the most general result is the sufficient condition of A. Kolmogoroff [4], according to which the convergence of the series

$$(3) \qquad \sum_{k=1}^{\infty} \frac{b_k}{k^2}$$

implies that the sequence (1) obeys the strong law of large numbers. This condition cannot be sharpened in the sense that, if for certain numbers b_k the series (3) diverges, it is always possible to find a sequence of independent random variables with the dispersions equal to these b_k, which does not obey the strong

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