THE MOMENT PROBLEM OF ENUMERATING DISTRIBUTIONS

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By a distribution is meant any monotone function $\alpha = \alpha(x)$ satisfying $\alpha(-\infty) = 0, \alpha(\infty) = 1$ and $\alpha(x - 0) = \alpha(x)$. This implies that $\mu_0 = 1$, where

$$(1) \qquad \mu_n = \int_{-\infty}^{\infty} x^n \, d\alpha(x).$$

A given infinite sequence of real numbers $\mu_0 = 1, \mu_1, \mu_2, \cdots$ is the moment sequence (1) of at least one distribution α if and only if

$$(2) \qquad \det (\mu_{i+k})_n \geq 0, \qquad (n = 0, 1, 2, \cdots),$$

where $(a_{ik})_n$ denotes the n-th section of the infinity matrix (a_{ik}) in which $i, k = 0, 1, 2, \cdots$ (Grommer-Hamburger). If a distribution α is required which is constant for $-\infty < x < 0$, then (2) must be completed by the additional requirements

$$(3) \qquad \det (\mu_{i+k+1})_n \geq 0, \qquad (n = 0, 1, 2, \cdots),$$

(Stieltjes). If the x-set of increase is further restricted, further *infinite sequences of inequalities* must be required of the sequence $\mu_0, \mu_1, \cdots$ (this is illustrated by Hausdorff's case of "complete monotony", where the distribution α is required to be constant both for $-\infty < x < 0$ and for $1 < x < \infty$). Thus it seems to be of interest that if the x-set of increase is required to be contained in the sequence of all integers, then, under reasonable assumptions as to the determinate character of the moment problem, *a single numerical equality of simple type* takes the place of the respective infinite sequences of inequalities.

The same holds if the sequence of all integers is replaced by the sequence of all non-negative integers. A distribution α restricted in this latter manner may be called an *enumerating distribution.* This type is fundamental in the statistics of discrete events. In fact, if the possible states of a random variable are represented by a sequence of "urns" $U_0, U_1, \cdots$ and if λ_n is the probability that the random variable be in U_n, the distribution over the various "urns" is supplied by the step-function $\alpha = \alpha(x)$ having the jump λ_n at $x = n$, where $n = 0, 1, \cdots$ and $\lambda_0 + \lambda_1 + \cdots = 1$ (for instance, $\lambda_n = a^n e^{-a}/n\,!$ in Poisson's case, where a is a positive number determined by the standard deviation). The *instability* of the situation, expressed by the fact that a sharp *equality* replaces the *inequalities* of the Hausdorff type, explains to some extent the difficulties met before in such a connection [1].

Received July 28, 1944.