SOME APPLICATIONS OF THE FOURIER INTEGRAL TO GENERALIZED TRIGONOMETRIC SERIES

By Richard Bellman

1. **Introduction.** The present paper evolved out of an attempt to prove a theorem of Ingham [2; 374], a "high indices" theorem for Dirichlet series which is the following

THEOREM 1 (Ingham). *Suppose that the Dirichlet series*

$$f(u) = f(s + it) = \sum_1^\infty a_k e^{-l_k u} \tag{1.1}$$

and the function $f(u)$ defined by it satisfy the following conditions:

(1) $l_n - l_{n-1} \geq 2d > 0$;

(2) *the series is convergent for $s > 0$;*

(3) *for some fixed $T > \pi/2d$, the mean value*

$$\frac{1}{2T} \int_{-T}^{T} | f(s + it) |^2 \, dt$$

is bounded for $s \to +0$;

(4) *$f(u)$ is regular at $s = 0$, or $(f(u) - f(0))/u$ (suitably defined on $s = 0$) is continuous in the region $s \geq 0$, $-T \leq t \leq T$ for some $T > 0$.*

Then $\sum_1^\infty a_n$ converges to the sum $f(0)$.

This theorem constitutes simultaneously a generalization to and analogue for Dirichlet series of theorems of Fatou and M. Riesz on power series, and a theorem of Hardy-Littlewood, sometimes called the "high indices" theorem. The main difficulty of the theorem lies in proving that conditions (1), (2), (3) involve $a_n = o(1)$ as $n \to \infty$. We shall concern ourselves in the first section with deriving conditions ensuring this or the equally potent $a_n = O(1)$ as $n \to \infty$. We refer to Ingham's paper for the convergence proofs. We shall use the theory of Fourier integrals to derive in place of (3) a condition of an analogous type, but not comparable to it generally. Then using the apparatus at hand and the methods of this section, we shall prove a moment theorem for the sequence (e^{il_nt}), with the separation condition on the exponents, for the interval $(-\infty, \infty)$. Then, in passing, we show how a generalization of a Tauberian theorem of Wiener yields the Hausdorff-Young theorem for generalized trigonometric series. Finally we extend a theorem of Gorny connecting the coefficients of the expansion of a function in a generalized trigonometric series, satisfying a separation theorem, with the mean values of the derivatives of the function.

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