REGULAR CONVERGENCE

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In this paper we study the homology aspects of n-regular convergence, a type of convergence of sequences of closed sets introduced by Whyburn. By using some results on homology local connectedness which we proved elsewhere, we are able to obtain some new results on n-regular convergence and also to give new and simpler proofs of some known theorems.

Our chief result is in connection with a theorem due to White. He has considered the space L^n of all lc^n subsets of a compactum R, the topology in this space being defined in terms of n-regular convergence. Among other things, he has shown that this space is actually separable metric. We give a new proof of this theorem and also show that the space is topologically complete, thus introducing the possibility of arguments involving Baire category.

Using the same methods we also prove some theorems on the homology properties of the limit of a sequence of lc^n sets which converges n-regularly. In particular, we extend to the case of generalized manifolds of any dimensions a theorem of Whyburn's which states that the limit of a sequence of n-spheres, $n = 1$ or 2, which converges $(n - 1)$-regularly is an n-sphere.

In the following the compact metric space R is assumed, for convenience, to be of diameter 1. If P and Q are closed subsets of R, we denote by $\rho_H(P, Q)$ the Hausdorff distance between P and Q [1; 115]. Vietoris cycles are denoted by capital letters and finite chains and cycles by small letters. The coefficient domain can be any commutative ring which contains a unit element.

The following definition is due to Whyburn [11].

Definition 1. A sequence $\{P_i\}$ of closed subsets of R is said to *converge n-regularly* to a closed set P if $\lim \rho_H(P, P_i) = 0$ and if for each $\epsilon > 0$ there is an i_0 and a $\delta > 0$ such that if Γ is a cycle of dimension $\leq n$ and of diameter $< \delta$ in some P_i, $i \geq i_0$; then $\Gamma \sim 0$ in a subset of P_i of diameter $\leq \epsilon$.

Let us denote by L^n the collection of all closed lc^n (= locally connected in dimension 0 to n with respect to Vietoris cycles [3]) subsets of R. We turn L^n into a Fréchet limit space by: $\lim P_i = P$ if and only if the sequence $\{P_i\}$ converges n-regularly to P. In order to investigate this topology in L^n, we need another definition:

Definition 2. For each closed subset P of R and for each $\epsilon > 0$, let $\delta_n(\epsilon, P)$ be the least upper bound of all numbers δ such that any cycle in P of dimension n and of diameter $< \delta$ bounds in a subset of P of diameter $\leq \epsilon$. Let $\delta^n(\epsilon, P)$ be the minimum of $\delta_k(\epsilon, P)$ for $k = 0, 1, 2, \cdots, n$.

It is clear that for each P, $\delta^n(\epsilon, P)$ always exists and is a non-negative monotone non-decreasing, and hence measurable, function on the half-open interval $I^* = (0 < \epsilon \leq 1]$. If P is lc^n, then $\delta^n(\epsilon, P) > 0$ everywhere in I^* and conversely.

Received March 1, 1944.