SUFFICIENT CONDITIONS FOR A GENERALIZED-CURVE PROBLEM IN THE CALCULUS OF VARIATIONS

By Truman Botts

1. **Introduction.** *Generalized curves* were devised by L. C. Young [9], [10], [11] as a means of bringing within the scope of classical calculus of variations variational problems with unattained minima. Young showed that such problems may sometimes be reformulated with respect to an enlarged class of elements, called generalized curves, in such a way that a minimum unattained in the original class of curves *is* attained in the extended generalized-curve class, making classical methods applicable. In [11] Young investigated, for a class of generalized curves, the free problem in non-parametric form in the plane, and obtained for this problem correspondents of the DuBois-Reymond and Weierstrass conditions.

Young's extension was developed and applied to the study of existence theorems for Bolza problems by E. J. McShane [3], [4], [5]. McShane formulated for generalized curves the important notion of *vectors carried*; and, working with the problem of Bolza in parametric form with generalized curves, he derived in terms of this notion necessary conditions analogous to the Lagrange multiplier rule, the Weierstrass and Clebsch conditions, and the Dresden corner condition.

The present study constitutes, we believe, the first attempt to prove a sufficiency theorem for generalized curve problems. The problem considered is the free problem in non-parametric form in $(n + 1)$-space. The result obtained is not particularly strong: the sufficiency theorem arrived at guarantees only a "weak" relative minimum, and our definition of weak neighborhood is extremely restrictive. This order of restriction seems necessary, if there is to exist a second variation which behaves at all as in the ordinary curve case. However, as we show, it still permits the establishment of analogues of the DuBois-Reymond, Legendre, and Jacobi necessary conditions. Moreover, the sufficient conditions found are, with one exception, simply the familiar strengthenings of the corresponding necessary conditions.

Roughly, the paper falls into three parts. §§2, 3, 4, and 5 are expository, setting forth a description of generalized curves and their properties and of the minimum problem to be considered. §§6, 7, and 8 establish necessary conditions for this problem. §§9, 10, 11, and 12 are directly concerned with the sufficiency proof.

How heavily the present work depends on the published researches of Professor McShane will be evident to the reader of these pages. In addition, the author is indebted to Professor McShane for numerous helpful conversations, and particularly for the suggestion of the Stieltjes-derivation technique used in §11.

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