ORIENTED SYSTEMS

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In [2] I showed that a certain family of ordered systems is adequate for topological purposes in the sense that any monotone closure can be defined in terms of convergence of functions on this family of ordered systems. Each system of the family has two properties; the order relation is transitive and each element has a successor. If, in addition, such a system is non-empty, it is called oriented. The oriented systems include the directed systems (see §1) and every non-empty system of subsets of any set with inclusion as the order relation. It is the purpose of this paper to discuss certain equivalence and order relations between pairs of oriented systems and to begin the analysis of the structure of oriented systems.

Tukey [4], [5] showed that there are three equivalent ways of defining a partial order relation $\mathcal{S} > \mathcal{T}$ in the family of directed systems; one method involves the existence of certain functions, another depends on convergence in neighborhood spaces, and the third is described in terms of reordering. In §2 the first of these methods is applied to define an order relation $>$ for oriented systems and it is shown that most of the results for directed systems hold for oriented systems; in particular, Theorems (2.10) and (2.12) show that the relations between $>$, convergence and reordering still hold. The chief difference between the behavior of $>$ for directed and for oriented systems is the lack of a certain property; if $\mathcal{S}$ and $\mathcal{T}$ are directed systems such that $\mathcal{S} > \mathcal{T}$ and $\mathcal{T} > \mathcal{S}$, then $\mathcal{S}$ and $\mathcal{T}$ can be embedded as subsystems in a third directed system $\mathcal{R}$ so that each is cofinal in $\mathcal{R}$; i.e., for every element r or $\mathcal{R}$ there is an s of $\mathcal{S}$ and a t of $\mathcal{T}$ such that $s > r$ and $t > r$. To study when this property does hold and why it is true for directed systems, it is convenient to modify Tukey's approach to the problem by introducing a method analogous to one used by R. H. Fox [3] in the study of homotopy type and deformation retraction.

In §3 a second order relation $\succ$ among oriented systems is introduced; it is shown that this also agrees with Tukey's ordering for directed systems and that it is related to Alaoglu-Birkhoff convergence [1] in a neighborhood space in the same way that $>$ is related to the definition of convergence used in [2].

§4 discusses the structure of an oriented system in terms of its maximal directed subsystems. Two classes of oriented systems, the simple and the everywhere branching, are chosen as fundamental to the discussion; the principal result is the decomposition Theorem (4.8) which asserts that every oriented system has a cofinal subsystem which can be cut into two disjoint parts, one simple and one everywhere branching. The cardinal number $u(\mathcal{S})$ of distinct maximal directed subsystems of $\mathcal{S}$ is shown to give a rather precise measure of the "undirectedness" of the oriented system $\mathcal{S}$; its independence of the various

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