FUNCTIONS OF EXPONENTIAL TYPE, I

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1. It is well known [4; 362], [11; 286], [6; 68], [9] that an entire function of exponential type, periodic on the real axis, is a finite trigonometric sum. In this note the corresponding result is established when the function is almost periodic on the real axis—namely, the set of Fourier exponents is bounded, and in fact bounded by the type of the function. The kind of almost periodicity demanded is of little importance; all that is required in the proof is that $|f(x)|$ has a mean value. However, the proof would be almost trivial when $f(x)$ is bounded. To show that the apparently more general result is really more general, I construct an unbounded function of exponential type which is Besicovitch almost periodic on the real axis.

Let us write

$$M[f(x)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx.$$

The main theorem of this note is

THEOREM 1. *If $f(z)$ is an entire function of exponential type c (i.e., $|f(z)| < Ae^{c|z|}$), such that $M[|f(x)|]$ exists, then for $|\lambda| > c$, $M[f(x)e^{i\lambda x}]$ exists and is zero.*

Since B^1 almost periodicity [1] implies that $M[|f(x)|]$ exists, our result on almost periodic functions follows.

A simple proof of the theorem for periodic functions can be obtained by integrating by parts in the integral defining the Fourier coefficients of $f(x)$ and using S. Bernstein's inequality [10], $|f^{(n)}(x)| \leq c^n \max |f(x)|$.

2. To prove Theorem 1 we begin by showing that $f(x) = o(|x|)$ as $|x| \to \infty$. We need several lemmas.

LEMMA 1. *If $f(z)$ is an entire function of exponential type c, and*

$$(2.1) \qquad |f(x)| < \epsilon(|x|), \qquad \epsilon(x) = o(1) \qquad (x \to \infty),$$

then for $0 < \omega < c$

$$(2.2) \qquad |f(x + iy)| < \omega^{-1} e^{(c+\omega)|y|} \delta(|x|),$$

where $\delta(x) \downarrow 0$ as $x \to \infty$ and $\delta(x)$ depends only on $\epsilon(x)$.

We may suppose to begin with that $\epsilon(x) \downarrow 0$ (otherwise replace $\epsilon(x)$ by $\sup_{x \leq t} \epsilon(t)$) and that $\epsilon(x)$ is bounded. Since $f(z)$ is constant if $c = 0$ and (2.1) is satisfied, we suppose $c > 0$.

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