FUNDAMENTAL POLYNOMIALS OF LAGRANGE INTERPOLATION AND COEFFICIENTS OF MECHANICAL QUADRATURE

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1. **Introduction.** Some time ago, Dr. Paul Erdös communicated to me the following result due to Turán (still unpublished, to the best of my knowledge). Let

$$(1) \qquad x_{1,n} < x_{2,n} < \cdots < x_{n,n} \qquad (n = 1, 2, \cdots)$$

be a set of abscissas and set $\omega_n(x) = c(x - x_{1,n})(x - x_{2,n}) \cdots (x - x_{n,n})$, with c an arbitrary non-zero constant. Then

$$(2) \qquad l_{k,n}(x) = \frac{\omega_n(x)}{(x - x_{k,n})\omega_n'(x_{k,n})} \qquad (k = 1, 2, \cdots, n; n = 1, 2, \cdots)$$

are the fundamental polynomials of Lagrange interpolation of degree $n - 1$ corresponding to the n-th set of abscissas.

If the $\{x_{i,n}\}$ $(i = 1, 2, \cdots, n)$ *are the zeros of* $\cos(n \arccos x)$, *the Tchebycheff polynomial, and if* $x_{k,n} < x_0 < x_{k+1,n}$, *then*

$$| l_{1,n}(x_0) | < | l_{2,n}(x_0) | < \cdots < | l_{k,n}(x_0) |,$$

$$| l_{k+1,n}(x_0) | > | l_{k+2,n}(x_0) | > \cdots > | l_{n,n}(x_0) |.$$

The proof given by Turán, unfortunately from the standpoint of generalization, depended on the readiness with which $\omega_n'(x_{k,n})$ is computed for the Tchebycheff polynomial. In the present paper, we approach the problem in a totally different way, obtaining results which include that of Turán.

Several other successful efforts have been made to discuss properties of the fundamental polynomials of Lagrange interpolation chiefly in the case where (1) are the zeros of classical orthogonal polynomials. These efforts have been mainly in the direction of bounds for the moduli, or bounds for special sums. In the present paper, we make an investigation of relations among the set of fundamental polynomials corresponding to a particular n. The results are not always as satisfyingly simple as that of Turán. Nevertheless, they should serve to simplify efforts to find the best bounds for the moduli.

The means by which the results are obtained emphasize the intimate connection between Lagrange interpolation and Gauss mechanical quadrature in a way which does not seem to have been apparent heretofore. Also, the preliminary relations involving mechanical quadrature coefficients are of independent importance, as we hope to demonstrate at some later date.

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