REPRESENTATION AS A GAUSSIAN INTEGRAL

By Harry Pollard

1. **Introduction.** Various writers have studied the inversion and representation problems for the unilateral transform

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (x > 0)$$

by means of the Laguerre polynomials [1]. In particular, Widder has linked the completely monotonic case ($\alpha(t)$ non-decreasing) with the positivity of the Abel kernel for Laguerre series [5; 168-177].

A similar relation exists between the Hermite polynomials and the bilateral (Gaussian) transform

$$f(x) = \int_{-\infty}^\infty e^{-\frac{1}{2}t^2} e^{xt} d\beta(t). \tag{1.1}$$

The case in which $\beta(t)$ is an integral has been studied in considerable detail [2], [3]. In the present paper we shall treat the representation problem for non-decreasing $\beta(t)$ by a method paralleling Widder's, but we shall employ the Abel kernel for Hermitian series. Technical difficulties which are not encountered in the unilateral case arise here, chiefly because the exponential function is unbounded over $-\infty < x < \infty$.

Our principal result is the following.

Theorem 1.1. *In order that $f(x)$ have the representation* (1.1) *over $-a < x < a$, with $\beta(t)$ non-decreasing and bounded, it is necessary and sufficient that*

(i) *$f(x)$ be analytic in $-a < x < a$,*

(ii) $\sum_{i=0}^n \sum_{j=0}^n f^{(i+j)}(0)\xi_i\xi_j \geq 0$ *for all ξ_i and all $n \geq 0$,*

(iii) *the series* $A_0 = \sum_{n=0}^\infty \frac{f^{(2n)}(0)}{2^n n!}$ *converge.*

The necessity of the conditions is obvious. A proof of the sufficiency can be derived from Hamburger's criterion for the representation of a function as a bilateral Laplace transform [5; 268], but this method fails to show the connection with the Hermite polynomials. Our proof proceeds upon entirely different lines.

2. **The Hermite polynomials.** These polynomials are defined by

$$H_n(t) = \sum_{k=0}^n d_{kn} t^k = (-1)^n e^{t^2} [e^{-t^2}]^{(n)} \qquad (n = 0, 1, 2, \cdots), \tag{2.1}$$

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